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General Relativity and Its Effects on Planetary Orbits and Interplanetary Observations

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LINCOLN LABORATORY

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ABSTRACT

A discussion is presented of all aspects of the General Relativity Theory that can be tested at present or in the near future by performance of interplanetary radar experiments. The assumptions underlying general relativity and the mathematical concepts and techniques used are first introduced. Then the approximate equations of motion, to the accuracy needed for the solar system, are derived. Finally, the propagation of electromagnetic waves according to general relativity is considered, and expressions for the time delays and Doppler shifts are obtained.

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CONTENTS

GENERAL RELATIVITY AND ITS EFFECTS ON PLANETARY ORBITS AND INTERPLANETARY OBSERVATIONS

I. INTRODUCTION

New radar installations, such as the 7840-Mcps Haystack facility developed by the M.I.T. Lincoln Laboratory, will shortly provide much more data on the time delays and Doppler shifts of interplanetary radar signals. These data, when combined with other optical and radar observations, will allow some of the predictions of Einstein's General Relativity Theory (GRT) to be tested. Conventional astronomy assumes that the equations governing the motion of celestial bodies are those of Newtonian Gravitational Theory (NGT), and that those governing the propagation of electromagnetic signals are given by the Special Relativity Theory (SRT). The corresponding equations derived from GRT are conventional in the lowest approximation, but small correction terms appear in higher approximations; the tests will be performed to determine the presence or absence of these terms.

In this report, we first discuss the assumptions that underlie GRT and introduce the mathematical concepts and techniques that are to be used. We then obtain the approximate equations of motion to the accuracy needed for the solar system. Since there is a good deal of controversy among workers in the field with regard to how the approximate equations are to be obtained from the exact ones, the views and assumptions of the various schools are discussed at some length. All the approaches yield the same equations of motion for the solar system except for higher-order terms which are so small as to be unmeasurable. The propagation of electromagnetic waves according to GRT is then considered, and expressions for the time-delay and Doppler-shift corrections obtained.

II. THE FIELD EQUATIONS AND EQUATIONS OF MOTION

Let us begin by writing down the field equations of GRT and discussing the expressions in them:[1]

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik} \quad . \tag{2-1}$$

In explanation of the notation, it must be understood that we are dealing with functions in a four-dimensional space which has the property of being locally Minkowskian, i.e., we can, by using suitably chosen coordinates, make an infinitesimally small neighborhood around each point look like the usual space-time of special relativity. If we have an expression written down in one coordinate system, then the indices tell us how to rewrite it in another coordinate system, the general rule being[2]

$$\overline{A}^{j_1 \ldots j_M}_{i_1 \ldots i_N}(\overline{x}) = \frac{\partial \overline{x}^{j_1}}{\partial x^{k_1}} \cdots \frac{\partial \overline{x}^{j_M}}{\partial x^{k_M}} \frac{\partial x^{\ell_1}}{\partial \overline{x}^{i_1}} \cdots \frac{\partial x^{\ell_N}}{\partial \overline{x}^{i_N}} A^{k_1 \ldots k_M}_{\ell_1 \ldots \ell_N}(x) \tag{2-2}$$

where

$$\overline{x}^i = \overline{x}^i(x) \quad ; \quad \frac{\partial(x^1, x^2, x^3, x^4)}{\partial(\overline{x}^1, \overline{x}^2, \overline{x}^3, \overline{x}^4)} \neq 0 \tag{2-3}$$

where $\overline{x}^i$ and x^i are the coordinates of the same physical point in the two systems. In Eq. (2-2) and in the sequel we use the Einstein summation convention: A repeated latin index is to be summed from one to four. Objects transforming according to Eq. (2-2) are called tensors and because of the linearity of the transformation law, a tensor equation has the same form in all coordinate systems. Thus Eq. (2-1) is shorthand for a set of 16 equations and can be written down in any coordinate system.

SRT rests essentially on three postulates:[3]

(a) There is a class of privileged reference frames, called inertial frames, in which the laws of physics are the same.

(b) The velocity of a light signal appears to be the same when measured in each inertial frame.

(c) Space-time is homogeneous.

Translating this into mathematical language, we find that the Cartesian coordinates assigned to an event in space-time in different inertial frames are connected by Lorentz transformations, and, in order to have the equations retain their form, that basic equations should be tensor equations (in SR we do not deal with general nonsingular transformations of the form of Eq. (2-3) but only with Lorentz transformations). A Lorentz transformation is a linear transformation with the property that

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2(t_A - t_B)^2$$

$$= (\overline{x}_A - \overline{x}_B)^2 + (\overline{y}_A - \overline{y}_B)^2 + (\overline{z}_A - \overline{z}_B)^2 - c(\overline{t}_A - \overline{t}_B)^2 \tag{2-4}$$

if A and B are two events. By looking at two infinitesimally close events we can rewrite Eq. (2-4) as

$$ds^2 = -\eta_{ij} dx^i dx^j = -\bar{\eta}_{ij} d\bar{x}^i d\bar{x}^j = \overline{ds}^2 \tag{2-5}$$

where

$$\eta_{ij} = \text{diagonal}\,(1, 1, 1, -1) = \bar{\eta}_{ij} \quad ; \quad (\eta_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2-6}$$

and

$$x^1 = x \quad , \quad x^2 = y \quad , \quad x^3 = z \quad , \quad x^4 = ct \quad . \tag{2-7}$$

Therefore, Lorentz transformations are linear transformations which leave the quadratic form in Eq. (2-5) invariant.

We can also formulate Lorentz transformations in a general curvilinear coordinate system, provided we know the relationship between the general coordinates and a set of inertial Cartesian coordinates, by writing

$$x^i = x^i(X^i) \qquad (X^i \text{ are inertial Cartesian coordinates}) \tag{2-8}$$

in which case

$$ds^2 = -g_{ij} dx^i dx^j = -\eta_{ij} dX^i dX^j \tag{2-9}$$

$$g_{ij}(x) = \frac{\partial X^k}{\partial x^i} \frac{\partial X^\ell}{\partial x^j} \eta_{k\ell} \qquad \left(\text{or } \eta_{ij} = \frac{\partial x^k}{\partial X^i} \frac{\partial x^\ell}{\partial X^j} g_{k\ell} \right) \tag{2-10}$$

and all tensor equations can likewise be modified trivially. Finally, we note that, relative to an inertial system, a free particle moves in a straight line in space-time. Since a straight line can, in ordinary space, be characterized by a variational principle (the arc length between two points is shortest along the straight line connecting them), we might expect a similar result to hold in Minkowski space.[3] It turns out that if we demand

$$\delta \int_A^B ds = 0 \tag{2-11}$$

where A and B are two points on the particle's trajectory, and

$$ds^2 = -\eta_{ij} dX^i dX^j \tag{2-12}$$

and use the standard techniques of the calculus of variations, then we obtain as Euler-Lagrange equations

$$\frac{d^2 X^i}{ds^2} = 0 \quad . \tag{2-13}$$

If we transform to curvilinear coordinates and use Eq. (2-9) and the variational principle, then in place of Eq. (2-13) we obtain[4]

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{k\ell} \frac{dx^k}{ds} \frac{dx^\ell}{ds} = 0 \tag{2-14}$$

where

$$\Gamma^{i}_{k\ell} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^{\ell}} + \frac{\partial g_{m\ell}}{\partial x^{k}} - \frac{\partial g_{\ell k}}{\partial x^{m}} \right) \qquad (2\text{-}15)$$

and

$$g^{im} g_{mk} = \delta^{i}_{k} \quad ; \quad \delta^{i}_{k} = \text{diagonal}\,(1, 1, 1, 1) \quad . \qquad (2\text{-}16)$$

Note that if $g_{ik} = \eta_{ik}$ then Eq. (2-14) reduces to Eq. (2-13). A trajectory which satisfies Eq. (2-14) is called a geodesic. The $\Gamma^{i}_{k\ell}$ are called Christoffel symbols of the second kind; they are not quite tensors (if they were, they would vanish in all coordinate systems if they vanished in one) but their transformation properties are of no interest to us. If we want the path of a light signal, we cannot use ds as a parameter since ds = 0 along the trajectory; it can be shown that Eq. (2-14) can then be replaced by[4]

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^{i}_{k\ell} \frac{dx^k}{d\lambda} \frac{dx^\ell}{d\lambda} = 0 \quad ; \quad g_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} = 0 \quad . \qquad (2\text{-}17)$$

Equations (2-17) define a null geodesic; they determine the path and the parameter (called a "distinguished parameter along the geodesic").

GRT holds that if matter is present, particles still move along geodesics, but the g_{ij} no longer have the form of Eq. (2-10) so the geodesics are not straight lines. As stated above, we assume space-time is locally Minkowskian so that it is possible to introduce coordinates in terms of which

$$g_{ij}(P) = \eta_{ij} \qquad (2\text{-}18)$$

where P is any point, but in these coordinates $g_{ij}(Q) \neq \eta_{ij}$ in general if Q is not in an infinitesimal neighborhood of P. If it is possible to introduce a coordinate system in which $g_{ik} = \eta_{ik}$ globally the space is called flat. As we will see, the left-hand side of the field equations (2-1) vanishes for a flat space.

Given a set of sixteen functions $g_{ij}(x)$ with $g_{ij} = g_{ji}$ how can we tell if the space is flat? This is a problem in partial differential equations, since it is obviously equivalent to asking under what circumstances the equations

$$g_{ij}(x) = \frac{\partial X^1}{\partial x^i} \frac{\partial X^1}{\partial x^j} + \frac{\partial X^2}{\partial X^i} \frac{\partial X^2}{\partial x^j} + \frac{\partial X^3}{\partial x^i} \frac{\partial X^3}{\partial x^j} - \frac{\partial X^4}{\partial x^i} \frac{\partial X^4}{\partial x^j}$$

$$= \eta_{k\ell} \frac{\partial X^k}{\partial x^i} \frac{\partial X^\ell}{\partial x^j} \qquad [\text{Eq. (2-10)}]$$

have solutions for X^i. The problem is quite analogous to the question as to whether a given three-dimensional vector field can be written as the gradient of a scalar:

$$\sum_{\alpha=1}^{3} A_\alpha dx^\alpha \overset{?}{=} \sum_{\alpha=1}^{3} \frac{\partial f}{\partial x^\alpha} dx^\alpha \quad . \qquad (2\text{-}19)$$

The necessary and sufficient condition for Eq. (2-19) to be true is known from elementary vector analysis to be

$$\vec{\nabla} \times \vec{A} = 0 \quad \text{or} \quad C_{\alpha\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} = 0 \qquad (\alpha, \beta = 1, 2, 3) \quad . \tag{2-20}$$

The corresponding necessary and sufficient condition for Eq. (2-10) to be soluble is[5,6]

$$R_{ik\ell m} = 0 \tag{2-21}$$

and

$$R_{ik\ell m} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^\ell} + \frac{\partial^2 g_{k\ell}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{i\ell}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^\ell}\right)$$

$$+ g_{np}\left(\Gamma^n_{k\ell}\Gamma^p_{im} - \Gamma^n_{\ell m}\Gamma^k_{i\ell}\right) \quad . \tag{2-22}$$

$R_{ik\ell m}$ is called the Riemann or curvature tensor. The term "flat space" and "curvature" come from the study of two-dimensional surfaces in ordinary three-dimensional space. For coordinates on a curved surface, the two-dimensional Riemann tensor is proportional to the ordinary curvature (Gaussian curvature) of the surface.[7] The tensors appearing on the left-hand side of the field equations (2-1) are defined as[8]

$$R_{ik} = g^{\ell m} R_{\ell imk} \qquad \text{(Ricci tensor)} \tag{2-23}$$

$$R = g^{ik} R_{ik} \qquad \text{(curvature scalar)} \tag{2-24}$$

Because of the way it is defined, the Riemann tensor has a great deal of symmetry (it has only 20 independent components instead of $4^4 = 256$) and its derivatives also satisfy some identities (Bianchi identities). The situation is again analogous to the case of the curl of a vector in three dimensions where, from Eq. (2-20) it can be seen that

$$C_{\alpha\beta} = -C_{\beta\alpha} \tag{2-25}$$

and

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \tag{2-26}$$

with Eq. (2-26) being identically true, whether or not $\vec{\nabla} \times \vec{A}$ vanishes. If we define

$$G_{ik} \overset{\text{def}}{=} R_{ik} - \frac{1}{2} g_{ik} R \qquad \text{(Einstein tensor)} \tag{2-27}$$

and

$$G^{ik} \overset{\text{def}}{=} g^{i\ell} g^{km} G_{\ell m} \tag{2-28}$$

then the Bianchi identities take the form[9]

$$\frac{\partial G^{ik}}{\partial x^k} + \Gamma^i_{k\ell} G^{k\ell} + \Gamma^k_{k\ell} G^{i\ell} \overset{\text{def}}{=} G^{ik}_{\ ;k} \equiv 0 \quad . \tag{2-29}$$

It is very important to notice that Eq. (2-29) is identically true because of the way the Einstein tensor is related to the Riemann tensor. We accordingly conclude that a necessary condition for the field equations of GRT to be consistent is

$$T^{ik}_{;k} = 0 \tag{2-30}$$

where

$$T^{ik} = g^{i\ell} g^{km} T_{\ell m} \quad . \tag{2-31}$$

Note: It is conventional to consider two tensors which are related as in Eq. (2-31) to be the "same" tensor and to speak of lowering and raising indices by means of g_{ij} and g^{ij}, thus, e.g.,

$$T_{ij} = g_{ik} T^k_j = g_{ik} g_{j\ell} T^{k\ell} \quad .$$

The same basic symbol is used for tensors related in this manner.

T_{ik} is a generalization of the energy-momentum tensor of special relativity. In flat space, if we have a tensor which satisfies the condition

$$\frac{\partial T^{ik}}{\partial x^k} = 0 \tag{2-32}$$

where x^k are Cartesian coordinates, then

$$\int dx^1 dx^2 dx^3 dx^4 \, \frac{T^{ik}}{\partial x^k} = 0 \tag{2-33}$$

and integrating over all space between two times, we find

$$P^i(t_2) - P^i(t_1) = \frac{1}{c} \int dx^2 dx^3 dx^4 \, [T^{i1}(\infty, x^2, x^3, x^4)] + \ldots \tag{2-34}$$

$$P^i(t) \overset{\text{def}}{=} \frac{1}{c} \int dx^1 dx^2 dx^3 T^{i4}(x^1, x^2, x^3, t) \tag{2-35}$$

and

$$P^i \overset{\text{def}}{=} (\vec{P}, E/c) \tag{2-36}$$

so if T^{ik} vanishes sufficiently rapidly at spatial infinity, then the four quantities P_i are conserved. If a theory is based on a variational principle and is Lorentz-invariant, such a T^{ik} can always be found if no "external fields" are present.[10] This situation is reminiscent of the result of nonrelativistic particle dynamics, where the energy and momentum of a system of particles are conserved if no external forces act on it; it can be shown that it is consistent to interpret P^i as the energy-momentum vector of the relativistic field (the tensorial nature of T^{ik} will guarantee that the P_i transform like a four-vector under Lorentz-transformation), i.e., $\vec{P}$ is the momentum and E the energy.

Suppose we wish to discuss "particles." A "particle," by definition, has a location, no extension in space and is completely characterized by a parameter "m_o" called its rest mass, and by its "velocity." It has an energy and momentum related by

$$E^2 - c^2 \vec{P}^2 = m_o^2 c^4 \quad . \tag{2-37}$$

We are thus more or less forced to take for T_{ik} the form[11]

$$T^{ik} = m_o c \frac{dx^i}{ds}\frac{dx^k}{dt}\,\delta(\vec{r}-\vec{R}) \quad , \quad ds^2 = (1 - v^2/c^2)\,dt^2 \quad \text{(SRT)} \quad . \tag{2-38}$$

This leads to

$$P^i = (m_o\gamma\vec{v}, m_o c\gamma) \qquad \left(\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\right) \tag{2-39}$$

which is the familiar result of relativistic mechanics.

How seriously should one take the idea of "point" particles? In electrodynamics, the use of δ-functions leads to infinite self-energies and other calamities which have to be argued away by "renormalization"[12] and which make the theory formally inconsistent (this formal inconsistency may just be due to not using a sophisticated enough mathematical formalism for handling δ-like functions,[13] but no one really knows). On the other hand, for extended matter distributions, no simple form for T^{ik} is possible since extended matter must involve internal degrees of freedom ("rigid" bodies can only exist if there are forces which act instantaneously at a distance and this violates the postulates of SRT).[14] People who are unhappy about using δ-functions but do not want to include internal forces explicitly, usually just let

$$m_o\delta(\vec{r}-\vec{R}) \to \mu(\vec{r}) \tag{2-40}$$

where $\mu(\vec{r})$ is "concentrated" around $\vec{R}$, and take the δ-function limit wherever it makes sense.[15] This is called the energy-momentum tensor for "incoherent matter." In order to include the interactions between the different parts of an extended body, elasticity theory and thermodynamics must be used as a guide to determine the phenomenological form.[16,17] It is not clear how a more fundamental level than the macroscopic one can be reached with this approach, but on the other hand it may very well be that all classical theories are inherently macroscopic since quantum theory becomes important for elementary particles and for very small distances. For astronomical problems the various bodies are so far away that internal forces can be effectively ignored and the forces between celestial objects can be calculated as if there were point planets (provided "renormalization" takes place, since for self-interactions the particle approximation is invalid); in the language of electrodynamics, a multipole expansion is used, keeping only the monopole moment (the problem of incorporating higher moments, e.g., spinning particles, has not yet been solved on a nonphenomenological level).[18]

We now come to one of the most remarkable results in general relativity.[19] In 1927 Einstein and Grommer found that the equations of motion of an isolated test particle could be derived from the field equations. To obtain this we first rewrite Eq. (2-40) to make it a tensor under a general transformation; the change is trivial:[20]

$$T^{ik} = \frac{m_o c}{\sqrt{-g}}\frac{dx^i}{ds}\frac{dx^k}{dt}\,\delta(\vec{r}-\vec{R}) \tag{2-41}$$

where

$$g = \det|g_{ik}| \quad . \tag{2-42}$$

We then integrate Eq. (2-30) around a tube in space-time which encloses the world-line of the test particle and no other matter, although other matter may be present outside of the tube. The properties of the δ-function under integration then lead to the result that the particle obeys the geodetic equations of motion:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{k\ell}\frac{dx^\ell}{ds}\frac{dx^k}{ds} = 0 \quad . \tag{2-43}$$

There has been one bit of "swindling" though; T^{ik} as defined in Eq. (2-41) depends explicitly on g_{ij}, and if we have a particle we would expect it to produce a metric tensor which is singular at the particle (analogous to the Coulomb potential at a point charge). The product of a singular function and a δ-function must be defined more carefully and in fact may be impossible to define. It is for this reason that we call the particle a "test" particle and assume that it does not contribute to the field so that the coefficient of $\delta(\vec{r}-\vec{R})$ in Eq. (2-41) is well defined. "Test" particles make intuitive sense but may not make mathematical sense in a nonlinear theory like GRT where the superposition principle does not hold. Nevertheless one is usually forced to think in terms of test particles in order to relate the theory to pre-relativity physics.

How can we obtain Newtonian Gravitational Theory (NGT) from GRT?[21] We would expect NGT to apply when we can ignore the finite velocity of light both for test particles and for the sources of the field as well as the nonlinearity of the field ("weak field") since NGT is a linear theory. We therefore assume that it is possible to find a coordinate system in which the metric can be expanded into a series

$$g_{ij} = \eta_{ij} + \lambda \underset{1}{h}_{ij} + \lambda^2 \underset{2}{h}_{ij} + \ldots \tag{2-44}$$

where λ is a formal parameter, and we keep terms to order λ. We further assume that in this coordinate system the time dependence of g_{ij} is entirely due to the motion of the sources. This corresponds, intuitively, to assuming there is no "gravitational radiation" incident on the system which is therefore "isolated." (The term "gravitational radiation" is in quotes because there is a great deal of controversy over what is meant by radiation in the theory and what effect, if any, it would have on the motion of a system.)[22] Since the velocities of the sources are assumed small

$$\left|\frac{\partial}{\partial x^4} g_{ij}\right| = \left|\frac{1}{c}\frac{\partial g_{ij}}{\partial t}\right| \approx \left|\frac{\vec{v}_{source}}{c}\cdot\vec{\nabla}g_{ij}\right| \tag{2-45}$$

or

$$\text{"}\frac{\partial}{\partial x^4}\text{"} \approx \frac{v}{c}\,\text{"}\frac{\partial}{\partial x^\alpha}\text{"} << \text{"}\frac{\partial}{\partial x^\alpha}\text{"} \tag{2-46}$$

where v is some characteristic velocity associated with the matter. For the same reason the equations of motion of a test charge can be simplified by using

$$\left|\frac{d\vec{x}}{cdt}\right| \sim \frac{v}{c} << 1 = \left|\frac{dx^4}{cdt}\right| \quad . \tag{2-47}$$

The equations of motion become, approximately[23]

$$\frac{d^2\vec{x}}{dt^2} = +\frac{c^2}{2}\vec{\nabla}\lambda \underset{1}{h}_{44} \tag{2-48}$$

which is to be compared to Newton's Law

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\varphi \quad . \tag{2-49}$$

In order to find the relation between $\underset{1}{h}_{44}$ and the sources of the field, we use the T_{ik} due to incoherent matter and the small velocity approximation,

$$T_{44} \approx \rho >> \text{(other components)} \tag{2-50}$$

where ρ is the density of matter, and obtain[24]

$$\nabla^2 \lambda \underset{1}{h}_{44} = -\frac{8\pi}{c^2} G\rho \quad . \tag{2-51}$$

We also identify

$$g_{44} \cong -1 - \frac{2\varphi}{c^2} \tag{2-52}$$

$$\nabla^2 \varphi = +4\pi G\rho \tag{2-53}$$

which is just the equation for the Newtonian potential if G is the usual gravitational constant. The form of the results is most gratifying but it is very important to examine closely the extent to which we have derived Newtonian Mechanics. We therefore note the following points:

(a) We have not solved the field equations but, rather, the formally linearized equations obtained by writing $g_{ij} = \eta_{ij} + \lambda h_{ij}$ and dropping terms quadratic or higher in λ. It is not known to what extent there is a correspondence between solutions of the linearized equations and of the full equations. Solutions of the linearized equations can obviously be superposed to give a new solution, which is not generally true of the nonlinear full equations; on the other hand, the full equations are covariant under arbitrary coordinate transformations and the linearized ones are not.[25]

(b) NGT applies to bodies moving at arbitrary velocities in arbitrarily strong gravitational fields, and the basic equations (2-53) and (2-49) reflect this. There is nothing in these equations themselves that forces us to use solutions with v/c small or for which $|2\varphi/c^2| << 1$. Thus there are solutions of the approximate equations which violate the conditions required to derive the approximate equations from the full theory.

(c) For the N-body system the Newtonian potential, ignoring structure, is

$$\varphi(\vec{r}) = -G \sum_{i=1}^{N} \frac{m_i}{|\vec{r} - \vec{r}_i|} \tag{2-54}$$

This serves to define the "mass" of the i^{th} body in NGT which must, of course, be determined experimentally by fitting parameters to a solution, as must the gravitational constant. In the first approximation to GRT we end up with the same potential by identifying m_i with the integral over the body of the ρ in Eq. (2-53). In higher approximations, however, there will be, among other changes, corrections to φ which will make it nonlinear in the individual masses. Since

the "Newtonian" mass is determined as that value of a parameter which gives a best fit to a solution with φ in the form of Eq. (2-54), we may expect the masses determined from GRT to differ from the "Newtonian" values.

Before discussing higher approximations to the field equations, it is convenient to have on hand a particular exact solution of the field equations, the Schwarzschild solution. This solution is[26]

$$ds^2 = \left(\frac{1 - \frac{r_o}{4r}}{1 + \frac{r_o}{4r}}\right)^2 c^2dt^2 - \left(1 + \frac{r_o}{4r}\right)^4 (dr^2 + r^2d\Theta^2 + r^2 \sin^2 \Theta d\varphi^2) \qquad (2\text{-}55)$$

where

$$r_o = \frac{2Gm}{c^2} \quad . \qquad (2\text{-}56)$$

What physical situation does this correspond to? If we substitute the g_{ik} of Eq. (2-55) into the field equations (2-1) we find T_{ik} vanishes everywhere except at the point $r = 0$ when the g_{ik} and their derivatives are singular. Thus the solution represents matter at the origin. In fact, this solution could be obtained if the most general modification of the Minkowski metric in spherical coordinates were demanded, i.e.,

$$ds^2 = c^2dt^2 - (dr^2 + r^2d\Theta^2 + r^2 \sin^2 \Theta d\varphi^2) \qquad (2\text{-}57)$$

which would not destroy its spherical symmetry in space and its symmetry with relation to past and future time.[27] The general solution, up to a coordinate transformation, is that of Eq. (2-55). The quantity "r_o" appears as a constant of integration and its identification with the "mass" of the matter at the origin is made by examining the expression for ds^2 when $r >> r_o$,

$$ds^2 \approx \left(1 - \frac{r_o}{r}\right) c^2dt^2 - \left(1 + \frac{r_o}{r}\right)(dr^2 + r^2d\Theta^2 + r^2 \sin^2 \Theta d\varphi^2) \qquad (2\text{-}58)$$

and appealing to the correspondence with the Newtonian potential as given by Eq. (2-52). It is therefore said that the Schwarzschild solution represents the field of a structureless point mass fixed at the origin.

One must not be misled by the fact that we use the same symbols for the coordinates, viz. (r, Θ, φ, t), in the Minkowski metric, in the Schwarzschild metric, and in the Newtonian equations of motion. The Schwarzschild metric is that of a non-Euclidean space and there is no quantity which has all the properties of the Euclidean radius vector (i.e., being equal both to the distance from the center and to the length of the circumference of a circle divided by 2π).[27] By changing the coordinates in which we express the Schwarzschild solution, we can produce an infinite number of different "radius vectors," each of which can be identified with the Newtonian one far away from the source; all we must guarantee is that the asymptotic g_{ij} be of the form

$$g_{44} \sim -1 - \frac{A}{r} \quad , \quad g_{\alpha\beta} \sim \delta_{\alpha\beta} + 0\left(\frac{1}{r}\right) \qquad (2\text{-}59)$$

and that the relationship between the old and the new variables be in some sense "smooth." The whole question of which coordinates to use to express the Schwarzschild solution is a thorny one.

The coordinates used in Eq. (2-55) are the so-called "isotropic" ones[26] and it will be noted that $g_{44} = 0$ for $r = r_o/4$, the so-called "Schwarzschild radius." There is a strong temptation to ignore this since for all macroscopic bodies r_o lies well in the interior where it would not be expected that the Schwarzschild solution would be a good approximation to the real solution; it is, however, at least esthetically disturbing. It has been found that the peculiar behavior at $r_o/4$ is due to the choice of coordinates and can be eliminated by using different ones,[27] but then other strange things happen to light signals which penetrate the barrier. How much of this is mathematics and how much is physics is not known as yet.[28]

The form of the equations of the geodesics will be different for each choice of "r," but, provided that Eq. (2-59) holds and that no spurious zeros or infinities are introduced by the change of variables, the force will look like a "Newtonian" $1/r^2$ one plus additional terms which will be small if "r" is identified with the "Newtonian" radial variable. As Eddington[29] points out, GRT makes perfectly definite predictions; the ambiguity lies in Newtonian theory which assumes that the force is strictly $1/r^2$ in nature. In GRT, "r" is just a marker. If we just use the asymptotic form (taking the effects of planet-planet interactions into account in a manner that amounts to using Newton's Law according to Eq. (2-52), then we have exactly the usual Newtonian mechanics. The extra terms for the sun's field produce the famous advance of the perihelia of the planets. The precession of an almost closed orbit clearly has an observable meaning which is relatively independent of the choice of "r," so different "r"s give the same approximate formulas. This mixture of GRT and NGT is, as is well known, in very good agreement with observation.

This ambiguity in "r," about which much has been written (much of it wrong!) is in fact to be expected and should not bother us unduly. As Bergmann[25] points out, the fact that the equations of GRT look the same in any coordinate system is not just a minor generalization on equations looking the same in any Galilean or Minkowskian inertial frame. Mathematically the difference is between invariance under an infinite dimensional Lie group and invariance under a finite dimensional Lie group. The analogue of general invariance is gauge invariance in electromagnetic theory, the coordinates in GRT being the analogues of the potentials in electrodynamics. Given the electrodynamic potentials, we certainly can calculate the fields, but since all we can observe in nature are gauge-invariant quantities, we never can determine the potentials experimentally.

Much work has gone into the problem of determining a "complete set of observables" for GRT, but little has been accomplished.[25] No one has observed "r" for a celestial body; what has been observed is light or radar reflected from a planet or, say, emitted by a comet. GRT tells us how this electromagnetic radiation propagates and how it is affected by its passage through a gravitational field. NGT makes no predictions along these lines. For different choices of "r," the behavior of light rays will be different but quantities like time delays have an unambiguous meaning since they measure the time it takes for a signal to travel from one point on the earth to the planet and back again. In different coordinate systems we would assign different values of the "markers" to the positions of the earth and of the planet relative to the sun, but the relationship between the assigned "markers" and the time delay is unambiguous and can be checked experimentally.

We shall discuss optics in the Schwarzschild field further in a later section of this report. Let us now turn to the problem of the equations of motion for an N-body system. If the solar system is considered isolated, then the usual Newtonian equations determine the motion of the

planets relative to the sun if these relative positions and the corresponding relative velocities are known at any one time. Therefore, what we ideally want in GRT is differential equations which give us the acceleration of each body (specified by some parameters) in terms of the relative positions and velocities of the other bodies. By the very nature of a relativistic theory such hopes are doomed except in the limit in which the velocity of light is infinite, i.e., in the "lowest" approximation. There are three reasons for this:

(a) Retardation. Interactions between particles take place via fields which propagate at the speed of light. Thus even if forces can be written to depend on interparticle distances and velocities, they must refer to retarded quantities, i.e., to the values occurring at an earlier time such that signals emitted then will arrive at the present moment at the particle of interest. In order to express everything in terms of variables at a single time, it is necessary to make a power series expansion in the retardation time and cut it off before higher derivatives than the velocity appear. There are further complications since at any given time some of the energy and momentum of the system is "in transit" so that, for example, the momentum of the system, which is approximately conserved, is not just the sum of the momenta of the particles, and the expression for the center of energy (the analogue of the center of mass) becomes much more complicated than is the case in Newtonian theory.

(b) Proper Time. Relativistic equations of motion such as the geodesic equation (2-14), always involve derivatives with respect to proper time along the world line of the particle. To obtain differential equations in terms of coordinate time, ds/dt must be expanded. This involves fixing a coordinate system and the resulting expressions will involve the "absolute" velocities of all the particles in this system rather than just the relative velocities of the particles. There seems to be no way of avoiding this. In the standard treatment of the GRT advance of perihelion, the fields corresponding to the sun at rest are used so the absolute velocities of planets become velocities relative to the sun. This, however, ignores the forces exerted by the planets. If we try to include these forces then the metric becomes time-dependent and we can no longer exactly identify coordinate time with proper time at the sun, i.e., the "time" in the static Schwarzschild solution, and so we run into the same difficulties.

(c) Radiation. This is a basically unsolved problem. If we have a system of charged particles then we expect them to radiate so that some kind of "radiation reaction" force must be included in addition to the interparticle forces. Since the radiation reaction force involves the time derivative of the acceleration, it radically changes our concept of equations of motion and in fact converts them from differential to integro-differential equations whose properties are not too well understood.[30] Fortunately, since we are forced for retardation reasons to expand in powers of $1/c$ anyhow, we can avoid the radiation reaction force in the first two approximations[31] because the radiation reaction force is proportional to $1/c^3$. Thus, if we are careful to keep only terms up to $1/c^2$ we can "forget" about radiation. In electrodynamics, in the lowest approximation we obtain the Coulomb interactions only, while in the second approximation we get the so-called Darwin Hamiltonian, run into absolute velocities, and in fact get very similar results to those we will obtain in the post-Newtonian approximation to GRT.

A careful discussion of all this, including an analysis of the Lorentz covariance of the resulting theory (it is covariant with respect to Lorentz transformations if we formally expand the transformation equations in powers of $1/c$ and keep terms only up to $1/c^2$) can be found in Fock's monograph.[32] Sweeping radiation under the rug in this manner is fine formally, but a skeptic might ask why one believes the approximate theory bears any resemblance to the full theory. The only reply one can make is to express the pious hope that radiation by slowly moving charges is a small effect and can be ignored. Whether or not the Darwin Hamiltonian is a good approximation to the full theory, its quantum

theoretical analogue, the Breit Hamiltonian, seems to work very well for two-electron systems.[33] The situation in GRT is even worse since no one knows what radiation means or whether or not it makes sense to talk about radiation of energy and momentum by an isolated system.[22] If one sticks to the linearized theory there is a complete analogy to electrodynamics and radiation is well defined and goes as $1/c^5$ (Ref. 34) so one can again sweep it under the rug in the lowest few approximations. The difficulty is that no one has any idea of the amount of correspondence between the linearized and full theories. In any case, we can formally obtain the post-Newtonian equations of motion without running into trouble, but it is difficult to go beyond.[35]

We now discuss the various derivations of the post-Newtonian equations of motion that appear in the literature. All end up with the same equations although the derivations are superficially different. The common idea is to expand the field equations in powers of $1/c$ as we did in deriving the Newtonian equations. Then it is assumed that the energy-momentum tensor (1) looks like a sum of δ-functions corresponding to each particle or (2) represents incoherent matter with the matter "concentrated" near the location of each particle or (3) vanishes outside of a number of nonintersecting arbitrarily narrow tubes in space-time within which it becomes singular. Much verbal blood has been shed by proponents of the different energy-momentum tensors but it is obvious that they are all rather equivalent and so it is not surprising that they all get the same equations. Whether one uses δ-functions or "concentrated" matter is obviously irrelevant except that with δ-functions one has to drop some formally infinite terms which do not appear when the matter is merely "concentrated." The derivation based on solving the empty-space equations in a multiply-connected region outside the world-lines of the sources is quite different in form.

Consider first the δ-function approach.[20] We have seen that to the lowest order the only component of the metric tensor which is determined by comparison with Newtonian theory is

$$g_{44} = -1 - \frac{2\varphi}{c^2} \qquad \text{[Eq. (2-53)]}$$

where φ is the Newtonian potential. The other components are highly arbitrary[36] since if we are given

$$g_{ij} = \eta_{ij} + \lambda h_{ij} \qquad \text{[Eq. (2-45)]}$$

we can perform an infinitesimal coordinate change

$$\bar{x}^i = x^i + \lambda\xi^i(x) \tag{2-60}$$

with $\xi^i(x)$ constrained only to vanish at infinity and to satisfy

$$\frac{\partial\xi^i}{\partial x^4} = 0(\lambda)\,\frac{\partial\xi^i}{\partial x^\alpha} \tag{2-61}$$

which is necessary to preserve

$$\text{"}\frac{\partial}{\partial\bar{x}^4}\text{"} \ll \text{"}\frac{\partial}{\partial\vec{\bar{x}}}\text{"} \quad . \tag{2-62}$$

Thus both the barred and unbarred coordinate systems are equally suitable in the Newtonian limit, but the metric tensor in the barred system has components

$$\bar{g}_{ij} = g_{ij} - \lambda \left(\frac{\partial\xi_i}{\partial x^j} + \frac{\partial\xi_j}{\partial x^i}\right) + 0(\lambda^2) = \eta_{ij} + \lambda \left[h_{ij} - \left(\frac{\partial\xi_i}{\partial x^j} + \frac{\partial\xi_j}{\partial x^i}\right)\right] \tag{2-63}$$

$$\xi_i = \eta_{ij}\xi^j \quad . \tag{2-64}$$

Because of the restriction of Eq. (2-61),

$$\bar{g}_{44} = g_{44} + 0(\lambda^2) \tag{2-65}$$

but all the other $\bar{h}_{ij}$ are arbitrary although one must obviously choose some values for them in order to pin down the coordinate system to first order. It is conventional to choose $\bar{h}_{ij}$ to make the first approximation to the metric[20]

$$ds^2 = \left(1 + \frac{2\varphi}{c^2}\right) c^2dt^2 - \left(1 - \frac{2\varphi}{c^2}\right)(dx^2 + dy^2 + dz^2) \quad . \tag{2-66}$$

The historical basis for this choice of metric is that the equations of the linearized theory of gravitation then assume their simplest form.[37] This choice is almost completely analogous to that of the Lorentz Gauge for the potentials in electrodynamics, the coordinates in Eq. (2-66) being known as "harmonic" or "de Donder" coordinates. Except for Fock's school,[16,17] few attach any special significance to harmonic coordinates in the full theory (Fock's point of view is discussed on page 15). In any case, once the first approximation to h_{ij} has been chosen, the second can be developed.

Accordingly, we substitute the g_{ij} of Eq. (2-66) into

$$T^{ik} = \sum_a \frac{m_o c}{\sqrt{-g}} \frac{dx^i}{ds} \frac{dx^k}{dt} \delta(\vec{r} - \vec{R}_a) \qquad \text{[Eq. (2-42)]}$$

which yields, for example,[20]

$$T^{44} = \sum_a m_a c^2 \left(1 + \frac{5\varphi_a}{c^2} + \frac{v_a^2}{2c^2}\right) \delta(\vec{r} - \vec{R}_a) \tag{2-67}$$

and solve the resulting field equations. One solution correct to second order is[20]

$$g_{\alpha\beta} = + \left(1 - \frac{2\varphi}{c^2}\right) \delta_{\alpha\beta} \tag{2-68}$$

$$g_{\alpha 4} = -\frac{G}{2c^3} \sum_a \frac{m_a}{|\vec{r} - \vec{R}_a|} \left[7v_{a\alpha} + \vec{v}_a \cdot \frac{(\vec{r} - \vec{R}_a)(\vec{r} - \vec{R}_a)_\alpha}{|\vec{r} - \vec{R}_a|^2}\right] \tag{2-69}$$

$$-g_{44} = 1 + \frac{2\varphi}{c^2} + \frac{2\varphi^2}{c^4} - \frac{2G}{c^4} \sum_a \frac{m_a\varphi_a}{|\vec{r} - \vec{R}_a|} - \frac{3G}{c^4} \sum_a \frac{m_a v_a^2}{|\vec{r} - \vec{R}_a|} \quad . \tag{2-70}$$

In order to obtain the last three equations it is necessary to ignore all terms which become infinite at the positions of the particles. This can be "justified" by looking at "concentrated" matter instead of at "point masses." The equations of motion of a test particle which moves in such a field can now be obtained; they are just the equations of a geodesic and therefore follow from the variational principle

$$\delta \int ds = 0 \qquad \text{[Eq. (2-11)]}$$

which we may rewrite to introduce a Lagrangian

$$\delta \int L dt = 0 \tag{2-71}$$

$$L = -mc \frac{ds}{dt} \tag{2-72}$$

where the coefficient "$-mc$" is conventional. Although it is not necessary for our purposes, one can obtain a Lagrangian for the N-body system. This is not the sum of the Lagrangians for each body but is constructed so that it leads to the correct force on each body for a given motion of the other bodies. The result is[20]

$$L = \sum_a \frac{m_a v_a^2}{2}\left(1 + 3\sum_b \frac{Gm_b}{c^2 r_{ab}}\right) + \sum_a \frac{m_a v_a^4}{8c^2} + \sum_{a,b} \frac{Gm_a m_b}{2r_{ab}}$$

$$- \sum_{a,b} \frac{Gm_a m_b}{4c^2 r_{ab}} [7\vec{v}_a \cdot \vec{v}_b + (\vec{v}_a \cdot \hat{n}_{ab})(\vec{v}_b \cdot \hat{n}_{ab})] - \sum_{a,b,c} \frac{G^2 m_a m_b m_c}{2c^2 r_{ab} r_{ac}} \tag{2-73}$$

where

$$r_{ab} = |\vec{r}_a - \vec{r}_b| \quad , \quad r_{ab}\hat{n}_{ab} = \vec{r}_a - \vec{r}_b \tag{2-74}$$

and all terms in the sum which blow up are understood to be omitted.

In this way the GRT N-body problem can be reduced in second approximation to a problem in classical mechanics with the Lagrangian in Eq. (2-73) replacing the one of ordinary Newtonian theory. The Lagrangian belongs to the standard class of those which involve velocity-dependent forces and a Hamiltonian can be constructed, conservation laws derived, etc., in the usual way. Since we are just interested in the equations of motion, we will not discuss these points here, but an exhaustive analysis may be found in the monograph of Infeld and Plebanski.[38]

There is, however, one very important result which follows from the fact that the post-Newtonian equations of motion are derivable from the Lagrangian Eq. (2-73). In the discussion of the Newtonian approximation we saw that although the equations of motion were unique, the coordinate system was not. A similar result is true in the post-Newtonian approximation. The post-Newtonian g_{ij} [Eqs. (2-68) to (2-70)] are far from unique even with the asymptotic boundary condition; the general solution involves four arbitrary functions which go to zero sufficiently rapidly at spatial infinity. If one computes the Lagrangian which replaces Eq. (2-63) when the general solution is used, however, one finds that it differs from Eq. (2-73) by a total time derivative,[39] and it is well known from classical mechanics that this means that both Lagrangians lead to the same equations of motion.[39] It is not known whether similar results hold in higher orders because of the ambiguities involving gravitational radiation.

Before discussing the equations of motion themselves, let us examine in some detail the other derivations of the equations of motion which appear in the literature. We will first outline Fock's derivation,[16] which is straightforward, and then his motivations, which are controversial.[40] Fock starts with the field equations

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik} \qquad \text{[Eq. (2-1)]}$$

which by the definition of R in Eq. (2-24) can be written as

$$R_{ik} = \frac{8\pi G}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T\right) \quad . \tag{2-75}$$

He now decomposes R_{ik} into a sum of terms, none of which separately transform as tensors. Thus, symbolically,

$$R_{ik} = A_{ik} + B_{ik} = \frac{8\pi G}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T\right) \quad . \tag{2-76}$$

Fock finds that if a particular class of coordinate systems, the so-called "harmonic" or "de Donder," systems, are used, then the B_{ik} in Eq. (2-76) vanish. Thus the Einstein equations can be replaced by the equations

$$A_{ik} = \frac{8\pi G}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T\right) \tag{2-77}$$

$$C_i = 0 \tag{2-78}$$

where Eqs. (2-78) are four equations which must be satisfied by g_{ij} in order to make the metric "harmonic." Equations (2-77) and (2-78) are no longer "tensor" equations since A_{ik} is not a tensor, but are still a perfectly respectable set of partial differential equations. There is no longer a need to worry about matters such as the Bianchi identity when solving Eq. (2-77) but of course a solution of Eq. (2-77) is no longer necessarily a solution of the Einstein field equations; it is only a solution if it also satisfies Eq. (2-78), in which case of course it will satisfy the Bianchi identity in the harmonic coordinate system since A_{ik} will be the Ricci tensor. As usual, all this has an analogue in classical electrodynamics. Maxwell's equations can be written in terms of the four potentials. Given any set of potentials which satisfy the equations, any other potentials related to the originals by a gauge transformation yield the same fields. If an auxiliary condition is imposed on the potentials (e.g., the Lorentz condition) so as to simplify the equations, then the resulting equations for the potentials are, in the first place, not completely gauge invariant. In the second place, a solution of these equations is not in general a solution of Maxwell's equations but only becomes so if it satisfies the auxiliary condition as well. Roughly speaking, what Fock has done is to choose his coordinate condition to all orders, *ab initio*, rather than to choose it in each order as has been done in the derivations of the Newtonian and post-Newtonian equations outlined above. By solving Fock's equations in the Newtonian and post-Newtonian approximations, the end result is, as mentioned before, the exact same equations of motion.

Fock believes that Einstein's field equations are incomplete and that they should be supplemented with the requirement that only a special class of harmonic (or, in an iterative solution, approximately harmonic) coordinates be used because they are the most "physical." In order to clarify this, consider the case of a flat space-time. Suppose we decide to use spherical coordinates in place of Cartesian coordinates. Then the mathematics singles out the point at the origin of the spherical coordinate system (e.g., the transformation from Cartesian to spherical coordinates is singular at the origin since θ and φ are not well defined there) even though it is "physically" just another point in space time and geometrically indistinguishable from any other point.

Mathematically speaking, a flat space-time has a high degree of symmetry since every point is "equivalent" to every other point (homogeneity); and the fact that the g_{ij} are constants in Cartesian coordinates reflects this more clearly while any other coordinates will introduce "unphysical" singularities into the metric. This does not usually hamper physicists calculating in spherical coordinates since the singular properties of the origin can more or less automatically be taken into account by modifying boundary conditions (e.g., in quantum mechanics the wave function in Cartesian coordinates must be finite everywhere whereas solutions of the Dirac equation in spherical coordinates are allowed to be mildly singular at the origin). It is, however, more serious in GRT where the metric itself is being studied to separate "physical" from "coordinate" singularities.

It is found that if Fock's equations are solved for an empty space-time which is Minkowskian at spatial infinity, the resulting coordinates are just a set of Cartesian coordinates (or, to be more precise, the family of Cartesian coordinates related to each other by inhomogeneous Lorentz transformations). A similar theorem can be proved even if there is matter present providing a static solution (i.e., one for which the g_{ij} have no explicit time dependence) exists in which case Fock's coordinates become asymptotically Cartesian at spatial infinity. Fock wishes to generalize this and in particular to use only harmonic coordinates in discussing the problem of gravitational radiation. It is at this point that many other relativists part company with Fock, for if anything like gravitational radiation exists then there is no compelling reason to assume that space-time at infinity is asymptotically flat and therefore there is no compelling reason to consider solutions in harmonic coordinates as being freer of spurious "coordinate singularities" than others. Accordingly, most relativists feel that although harmonic coordinates may be convenient in particular problems where one is looking for a solution which is of the form of a static solution plus a small perturbation, they do not have any special significance in more general cases.

Finally, there is the Einstein-Infeld-Hoffman (EIH) method.[13,41,42] Like the other methods described above, it is based on a power series expansion in $1/c$, the assumption that the metric is asymptotically Minkowskian at spatial infinity, and the assumption that all time dependence is due to the motion of the particles. In its original formulation, the world-lines of the particles were assumed not to intersect, were each enclosed in a "tube" and the field equations over <u>all</u> space-time were then replaced by the equations, with no matter present, extended over the multiply-connected region of space-time outside the tubes.

One solution is obviously a flat space-time, i.e., one in which nothing is singular anywhere. Since this is what we expect if there is no matter <u>at all</u>, this solution is rejected as unphysical and solutions are sought which become singular on the world-lines of the particles. The surprising feature of GRT is that there are <u>no</u> solutions of the equations for arbitrary singularities and world-lines. This is a consequence of the Bianchi identities. To see how this comes about, we again turn to an analogy from electrodynamics.[43] Suppose that the free-field Maxwell equations hold everywhere outside the world-lines of charged point particles and that these world-lines do not intersect. The charge on one of these particles at a certain time can then be defined by Gauss' law as

$$q(t) = \frac{1}{4\pi} \oint \vec{E} \cdot d\vec{S} \tag{2-79}$$

where the integral is extended over a closed spatial surface surrounding the particle at this time and does not surround any other particles (this can be done because of the nonintersection hypothesis). Then, making use of Maxwell's equations in free space and Stokes' theorem yields

$$\frac{dq}{dt} = \frac{1}{4\pi} \oint \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S} = \frac{c}{4\pi} \oint \vec{\nabla} \times \vec{B} \cdot d\vec{S} = 0 \quad . \tag{2-80}$$

Thus there can be no solutions to Maxwell's equations in which the charges of isolated particles change. This is of course just a special case of the well-known result that one can derive the equation of continuity from Maxwell's equations and that therefore specified charge and current distributions must be consistent with the equation of continuity or else there are no solutions to Maxwell's equations.

There is one further important point to make. Suppose the particle does not carry a net charge but is a dipole. We can visualize this dipole as consisting of two equal and opposite charges which are so "close to each other" that the surface surrounding the particle always includes both charges. Then the integrals in Eqs. (2-79) and (2-80) are always zero and solutions can be found to Maxwell's equations in which the strength of the singularity, i.e., its dipole moment, varies in time (imagine the two charges exchanging charge). So the equation of continuity by itself imposes a restriction only on a part of the possible singularity. Since the electric field of a point charge is spherically symmetric and is an inverse square law force while the field of a dipole or higher multipole is not spherically symmetric and is an inverse cube (or higher) law force, the nature of the singularity can be identified by looking at how singular $\vec{E}$ becomes as the size of the tube decreases to zero. The multipole structure of the singularity can be "identified," and it can be said that, e.g., only a charge or only a dipole exists.

Obviously, this analysis depends critically on the linearity of Maxwell's equations. The analogues of the equation of continuity in GRT are the four Bianchi identities which act as consistency conditions and certainly impose four constraints on the world-lines of singularities. The fact that the theory is nonlinear and has the same form in any coordinate system wreaks havoc with the multipole expansion; in fact, the identification of singularities by means of the dependence of the fields on the "size" of the world tube as it shrinks to a line can only be done in low orders of approximation and it has so far proved impossible to characterize the properties of mass distributions in an invariant manner.[44] A not entirely independent problem that also arises is directly due to the nonlinearity of the equations which results in singularities being piled on top of singularities at each level of approximation. These can be argued away in the lowest orders by an averaging process (called "tweedling" by Infeld)[13,45] but again this is a non-invariant procedure.

In the original EIH paper, no assumptions were made at the outset about the nature of the singularities, but it was found, not unexpectedly, that assumptions had to be made about how singular the fields would be on the surface of the tube[42] and EIH assumed the least nontrivial singularity. Infeld[11] showed that this was equivalent, at least up to the post-Newtonial approximation, to assuming a δ-function distribution of matter of the form of Eq. (2-41) and that by using Eq. (2-41) from the beginning the calculation could be simplified tremendously.

We will now outline the essential features of Infeld's calculation. An expansion is made of g_{ij} as in Eq. (2-43) with $1/c$ as the parameter of smallness, and of the energy momentum tensor of Eq. (2-41). If all the terms of a given order are then collected, the field equations are replaced by an infinite set of coupled equations each of which has the semi-symbolic form

$$\frac{\partial K^{i\ell mn}}{\partial x^{\ell}\,\partial x^{n}} + N(g^{im}) + aT^{im} = 0 \quad . \tag{2-81}$$

Both the quantity $N(g^{im})$, which represents the terms of a given order which arise from multiplying terms of lower order as required by the nonlinearity of the equations, and the terms T^{im} which come from the energy momentum tensor, involve only quantities of lower order. Thus the only new h_{ij} appear in the first term. K is linear in these quantities and in fact looks just like the linear part of the Riemann tensor of Eq. (2-22). This is very useful for knowing just how much freedom there is in choosing a coordinate system, up to this order, which will simplify the equations. It also follows that Eq. (2-81) is consistent only if N + aT satisfies Bianchi-like identities with K regarded as a Riemann-like tensor. This leads to constraints on the motion of the sources which turn out to be, successively, the Newtonian and post-Newtonian equations. In higher orders, difficulties arise because some of the terms in N do not go to zero fast enough at spatial infinity and so the whole program must be re-examined.[35]

To summarize the state of the art, the physically reasonable assumptions that the gravitational field produced by a system of structureless point particles should vanish at infinity, have a time dependence only through the time dependence of the positions of the particles, and become singular at the position of the particle in a "spherically symmetrical" manner, lead to unambiguous equations of motion up to and including the post-Newtonian approximation (if one makes the additional assumption that the kinetic energy of each particle is of the same order of magnitude as the Newtonian potential energy, this last assumption leads to equations of motion which are symmetric in each of the particles). It is not at all clear to what extent this procedure can be extended beyond the post-Newtonian approximation or whether it is in any sense a "good" approximation to the exact solution from a mathematical point of view. There are further difficulties involved in extending the analysis to include particles with structure, e.g., spinning particles.[44]

III. THE POST-NEWTONIAN EQUATIONS OF MOTION

The post-Newtonian equations of motion can be derived in the standard manner from the Lagrangian of Eq. (2-73) by calculating the left-hand side of the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}_a}\right) - \frac{\partial L}{\partial \vec{X}_a} = 0 \tag{3-1}$$

where $\vec{X}_a$ and $\vec{V}_a$ are respectively the position and velocity vectors of the a^{th} particle and where

$$\frac{d}{dt} f(\vec{X}_a, \vec{V}_a) \overset{def}{=} \sum_a \frac{\partial f}{\partial \vec{X}_a} \cdot \vec{V}_a + \sum_a \frac{\partial f}{\partial \vec{V}} \cdot \frac{d\vec{V}_a}{dt} \quad . \tag{3-2}$$

The Eqs. (3-1) will differ markedly in appearance from the Newtonian equations. Since the Lagrangian contains no higher order time derivatives than the first, the equations will still contain no higher order derivatives than the second and will, in fact, be linear in the accelerations. However, because of the complicated dependence of the Lagrangian on the positions and velocities of the particles and the fact that the time derivative is to be calculated according to Eq. (3-2), <u>each</u> equation of the set will involve the acceleration of <u>all</u> of the particles. After a somewhat tedious calculation, the following equations of motion are obtained:

$$\begin{aligned}
\vec{a}_a + \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^3} &= -\vec{a}_a\left(3\sum_b \frac{m_b}{r_{ab}} + \frac{1}{2} v_a^2\right) - (\vec{V}_a \cdot \vec{a}_a)\vec{V}_a + \frac{7}{2}\sum_b \frac{m_b \vec{a}_b}{r_{ab}} \\
&+ \frac{1}{2}\sum_b m_b \frac{(\vec{X}_{ab} \cdot \vec{a}_b)\vec{X}_{ab}}{r_{ab}^3} + \sum_{bc} \frac{m_b m_c}{r_{ab}^3 r_{ac}} \vec{X}_{ab} \\
&+ \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^3} \sum_c \frac{m_c}{r_{bc}} + m_a \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^4} \\
&+ \sum_b m_b \frac{\vec{X}_{ab}}{r_{ab}^3}\left(-\frac{3}{2} V_a^2 - 2V_b^2 + 4\vec{V}_a \cdot \vec{V}_b + \frac{3}{2}\frac{(\vec{V}_b \cdot \vec{X}_{ab})^2}{r_{ab}^2}\right) \\
&+ \sum_b \frac{m_b}{r_{ab}^3}[\vec{V}_a(3\vec{V}_a \cdot \vec{X}_{ab} - 3\vec{V}_b \cdot \vec{X}_{ab}) + \vec{V}_b(-4\vec{V}_a \cdot \vec{X}_{ab} \\
&+ 3\vec{V}_b \cdot \vec{X}_{ab})]
\end{aligned} \tag{3-3}$$

where

$$\vec{X}_a \overset{def}{=} \text{position vector of } a^{th} \text{ particle} \qquad \vec{X}_{ab} \overset{def}{=} \vec{X}_a - \vec{X}_b$$

$$\vec{V}_a \overset{def}{=} \frac{d\vec{X}_a}{dt} \qquad r_{ab} \overset{def}{=} |\vec{X}_a - \vec{X}_b|$$

$$\vec{a}_a \overset{def}{=} \frac{d^2\vec{X}_a}{dt^2} \tag{3-4}$$

and the sums are extended over all the particles but the a^{th}. We have further simplified the equations by using units in which the speed of light and the gravitational constant are unity, i.e., in Eq. (3-3),

$$"v" = v/c \tag{3-5}$$

and

$$"m" = GM/c^2 \quad . \tag{3-6}$$

We are not really interested in the general solution of Eq. (3-3) but only in one that looks like a perturbed Newtonian solution. We are therefore justified in substituting the Newtonian values of the acceleration in those terms on the right-hand side of Eq. (3-3) which contain the accelerations explicitly, since the accelerations are multiplied by factors of the formal order of v^2/c^2 or the Newtonian gravitational potential, and, from the derivation of the post-Newtonian approximation as discussed in Sec. I, any errors incurred by this substitution are then of the same formal order as terms which have already been neglected. If we substitute

$$\vec{a}_a \approx -\sum_b \frac{m_b \vec{x}_{ab}}{r_{ab}^3} \tag{3-7}$$

on the right-hand side and reorder the terms slightly, we obtain

$$\begin{aligned}
\vec{a}_a + \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^3} &= 4 \sum_{bc} \frac{m_b m_c \vec{X}_{ab}}{r_{ab}^3 r_{ac}} + 4\vec{V}_a \sum_b \frac{m_b}{r_{ab}^3} \vec{V}_a \cdot \vec{X}_{ab} \\
&\quad - \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^3} \left[V_a^2 + 2V_b^2 - \frac{3}{2} \frac{(\vec{V}_b \cdot \vec{X}_{ab})^2}{r_{ab}^2} - \vec{a}_b \cdot \vec{x}_{ab} - 4\vec{V}_a \cdot \vec{V}_b \right] \\
&\quad + \frac{7}{2} \sum_b \frac{m_b \vec{a}_b}{r_{ab}} + \sum_b \frac{m_b}{r_{ab}^3} [-3\vec{V}_a(\vec{V}_b \cdot \vec{X}_{ab}) + \vec{V}_b(3\vec{V}_b \cdot \vec{X}_{ab} - 4\vec{V}_a \cdot \vec{X}_{ab})] \\
&\quad + \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^3} \sum_c \frac{m_c}{r_{bc}} + m_a \sum_b \frac{m_b \vec{X}_{ab}}{r_{ab}^4} \quad .
\end{aligned} \tag{3-8}$$

Equations (3-8) were first obtained by Eddington and Clark in 1938.[46] Substantially the same equations had been obtained in 1915 by Droste and De Sitter[47] who unfortunately made a mistake and so obtained the wrong coefficients for the last two terms in Eqs. (3-8). We will apply these equations to find the GRT corrections to the planetary equations of motion and to the earth-moon-sun system.

IV. GRT AND THE MOTION OF THE PLANETS

Although the equations of motion are symmetric in each of the bodies, there is an inherent asymmetry in the solar system due to the fact that the sun is so much more massive than the combined mass of all the other bodies. Thus, terms of the same formal order in $1/c$ can have vastly different numerical values depending on whether a solar or planetary mass is involved. As a first step, let us rewrite Eqs. (3-8) for the i^{th} planet to bring out the dependence on the solar mass.

$$
\begin{aligned}
\vec{a}_i + \frac{M\vec{X}_{io}}{r_{io}^3} + \sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3} = {} & \frac{4M^2\vec{X}_{io}}{r_{io}^4} + \frac{4M\vec{X}_{io}}{r_{io}^3}\sum_j \frac{m_j}{r_{ij}} + \frac{4M}{r_{io}}\sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3} \\
& - \frac{M\vec{X}_{io}}{r_{io}^3}\left[V_i^2 + 2V_o^2 - \frac{3}{2}\frac{(\vec{V}_o\cdot\vec{X}_{io})^2}{r_{io}^2} - \vec{a}_o\cdot\vec{X}_{io}\right. \\
& \left. - 4\vec{V}_i\cdot\vec{V}_o\right] - \sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3}\left[V_i^2 + 2V_j^2 - \frac{3}{2}\frac{(\vec{V}_j\cdot\vec{X}_{ij})^2}{r_{ij}^2}\right. \\
& \left. - \vec{a}_j\cdot\vec{X}_{ij} - 4\vec{V}_i\cdot\vec{V}_j\right] + \frac{7}{2}\frac{M\vec{a}_o}{r_{io}} + \frac{7}{2}\sum_j \frac{m_j\vec{a}_j}{r_{ij}} \\
& + \frac{M}{r_{io}^3}\left[-3\vec{V}_i(\vec{V}_o\cdot\vec{X}_{io}) + \vec{V}_o(3\vec{V}_o\cdot\vec{X}_{io} - 4\vec{V}_i\cdot\vec{X}_{io})\right] \\
& + \sum_j \frac{m_j}{r_{ij}^3}\left[-3\vec{V}_i(\vec{V}_j\cdot\vec{X}_{ij}) + \vec{V}_j(3\vec{V}_j\cdot\vec{X}_{ij} - 4\vec{V}_i\cdot\vec{X}_{ij})\right] \\
& + \frac{M\vec{X}_{io}}{r_{io}^3}\sum_j \frac{m_j}{r_{jo}} + M\sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3 r_{jo}} + \sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3}\sum_k \frac{m_k}{r_{ij}} \\
& + \frac{m_i M\vec{X}_{io}}{r_{io}^4} + m_i\sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^4} \qquad (4\text{-}1)
\end{aligned}
$$

where now the subscript "o" refers to the sun and the solar term is excluded from all sums. A similar equation can be written for the sun:

$$\vec{a}_o + \sum_i \frac{m_i \vec{X}_{oi}}{r_{oi}^3} = 4 \sum_{ij} \frac{m_i m_j \vec{X}_{oi}}{r_{oi}^3 r_{oj}} + 4\vec{V}_o \sum_i \frac{m_i}{r_{oi}^3} \vec{V}_o \cdot \vec{X}_{oi} - \sum_i \frac{m_i \vec{X}_{oi}}{r_{oi}^3} \left[V_o^2 + 2V_i^2 \right.$$

$$\left. - \frac{3}{2} \frac{(\vec{V}_i \cdot \vec{X}_{oi})^2}{r_{oi}^2} - \vec{a}_i \cdot \vec{X}_{oi} - 4\vec{V}_o \cdot \vec{V}_i \right] + \frac{7}{2} \sum_i \frac{m_i \vec{a}_i}{r_{oi}}$$

$$+ \sum_i \frac{m_i}{r_{oi}^3} [-3\vec{V}_o(\vec{V}_i \cdot \vec{X}_{oi}) + \vec{V}_i(3\vec{V}_i \cdot \vec{X}_{oi} - 4\vec{V}_o \cdot \vec{X}_{oi})]$$

$$+ \sum_i \frac{m_i \vec{X}_{oi}}{r_{oi}^3} \sum_j \frac{m_j}{r_{ij}} + M \sum_i \frac{m_i \vec{X}_{oi}}{r_{oi}^4} \tag{4-2}$$

where the sums run over all the planets.

Now, Eqs. (4-1) and (4-2) involve more quantities than we can measure since we must choose an origin for our coordinate system. If we look at the Newtonian terms alone, i.e., ignore the right-hand side of Eqs. (4-1) and (4-2), and subtract Eq. (4-2) from (4-1) we obtain the system of equations

$$\frac{d^2\vec{X}_{io}}{dt^2} + \frac{\vec{X}_{io}}{r_{io}}(M + m_i) + \sum_{j \neq i} m_j \left(\frac{\vec{X}_{ij}}{r_{ij}^3} + \frac{\vec{X}_{jo}}{r_{jo}^3} \right) = 0 \quad . \tag{4-3}$$

Thus, in the Newtonian approximation, we can obtain a set of equations, the "planetary equations," involving only relative quantities. Furthermore, by multiplying each Eq. (4-1) by m_i and Eq. (4-2) by M and adding all the equations together in the Newtonian approximation, we obtain

$$\sum_i m_i \vec{a}_i + M\vec{a}_o = \sum m_i(\vec{a}_i - \vec{a}_o) + \left(M + \sum_i m_i \right) \vec{a}_o = 0 \tag{4-4}$$

or

$$\frac{d}{dt} \left[\sum_i m_i \frac{d\vec{X}_{io}}{dt} + \left(M + \sum_i m_i \right) \frac{d\vec{X}_o}{dt} \right] = 0 \quad . \tag{4-5}$$

The last equation states that the Newtonian center of mass is unaccelerated. Eqs. (4-3) and (4-5) are equivalent to Eqs. (4-1) and (4-2) in the Newtonian approximation. The center of mass equation, unlike the planetary equations (4-3), does not involve only relative quantities.

The planetary equations do not involve the velocity of the center of mass and look like a complete self-contained set of equations. This is illusory. The masses which appear are parameters which must be determined by fitting the observed motion to the predicted motion with arbitrary masses; to do this one must have a theory of light and of other electromagnetic phenomena. Newtonian mechanics does not include anything about light so additional postulates are needed. The usual postulate is that light travels in a straight line with the same speed "c" as is obtained in terrestrial measurements of the speed of light. Once this assumption is made, the velocity of the center of mass indeed becomes irrelevant.

When we look at the GRT equations, we see that although only relative positions appear, absolute velocities are involved (this is inherent in higher non-relativistic approximations to a relativistic theory as was discussed in Sec. I). Now, we know that ignoring the relativistic terms completely allows one to obtain an excellent fit to the motion of the planets around the sun and that the principal remaining discrepancies can be removed by adding on the GRT Schwarzschild field produced by the sun, treating the sun as if it were at rest. There will thus be a very good solution in which the "absolute" velocity of the sun is much smaller than the velocities of the planets relative to the sun. We will make the assumption that if

$$\vec{w} \overset{\text{def}}{=} \text{"absolute velocity" of the sun} \tag{4-6}$$

then

$$(M_{sun})\,(w) \lesssim (M_{planet})\,(\text{velocity of planet relative to sun}) \quad . \tag{4-7}$$

This assumption cannot be rigorously justified: an attempt can only be made to try to see whether the equations derived using the assumption describe the observed motions of the planets. It is known that $\vec{w}$ can be set equal to zero and good results still obtained, and that the estimate we use is compatible with this. Another heuristic reason for this estimate is that it can give the Newtonian center of mass a small velocity as can be seen from Eq. (4-5). Equation (4-5) is only approximately true when the full post-Newtonian equations are used, but Eddington and Clark have shown that there is no secular acceleration of the Newtonian center of mass for the two-body problem in the post-Newtonian approximation and it may well be conjectured that this is true for the many-body problem.

In the case of several bodies, it is known that there is a "center of energy" which moves with constant velocity and that the "center of energy" reduces to the Newtonian center of mass when we go from the post-Newtonian to the Newtonian approximation.[48] Fock[49] shows that for any reasonable estimates of the sizes of the terms, the distinction between the "center of energy" and the Newtonian center of mass can be ignored in the two-body problem. We will assume the same is true in general. This is also the traditional assumption made by De Sitter[47] who was the first to look at the implications of the many-body equations for the solar system. De Sitter even made the further assumption that the "absolute velocity" of the Newtonian center of mass vanished since he wanted the most "symmetric" metric at large distances from the solar system.

If the above discussion of "absolute velocities" and the like seems confused and not very convincing, it only reflects in the author's opinion, the confused and not very convincing state of the art. To obtain our equations we have had to assume that, in some coordinate system, the absolute velocities of our particles are small compared to the speed of light and we have set things up so that far away from the solar system space-time becomes flat. But of course if we go far enough away we come to other stars and regions where space-time is no longer flat. Thus our coordinate system is only valid in a region which is large compared to the solar system but small compared to interstellar distances, and our arguments about asymptotic behavior must be reinterpreted in the light of this limitation. It may be, for example, that the freedom to choose a coordinate condition in each order is not really there but that there exist preferred coordinate systems, e.g., if there is a "center of mass" of the universe, a coordinate system in which this point is fixed may be the one to use, and therefore we should not be too cavalier in our treatment of absolute velocities of bodies in the solar system.

These problems of course are not peculiar to relativistic astronomy; even in Newtonian mechanics the planetary equations should be modified if the solar system is accelerating relative to the "fixed stars." We can summarize the situation as follows: We have obtained equations of motion by assuming the solar system contains all the matter in the universe. In reality it does not and no one knows how to modify the theory to include this fact or to what extent the equations of motion and conservation laws of the theory are affected. So we are not breaking the rules of the game by making any "reasonable" assumption about the "absolute velocity" of the solar system and looking at the equations which result.

Since we assume the Newtonian center of mass is unaccelerated, we will work with Eq. (4-1) for the planetary accelerations and replace Eq. (4-2) for the acceleration of the sun by

$$\vec{a} = -\sum_i \frac{m_i}{M} \vec{a}_i \quad . \tag{4-8}$$

If Eqs. (4-1) are written as

$$\vec{a}_o = \vec{F}_i \tag{4-9}$$

which defines $\vec{F}_i$, then by subtracting Eq. (4-2) from Eq. (4-9) we obtain

$$\frac{d^2\vec{X}_{io}}{dt^2} = \vec{F}_i + \sum_{\text{all } j} \frac{m_j}{M} \vec{F}_j \quad . \tag{4-10}$$

We know we want a solution which represents a perturbed Newtonian one. We therefore replace Eq. (4-10) by a series of approximate equations obtained by collecting terms on the right which are of the same order. We determine the order of a term by noting that a typical velocity of a planet relative to the sun is about 30 km/sec. Since the velocity of light is about 3×10^5 km/sec, this gives $v_{io}/c \sim 10^{-4}$. This is also about the ratio of the mass of a planet to the mass of the sun. Finally, the velocities relative to the sun satisfy the condition

$$v_{io}^2 \sim \frac{M}{r_{io}} \qquad (c = G = 1) \tag{4-11}$$

which follows from the virial theorem for an inverse square force. We can accordingly classify accelerations as follows:

$$0^{th} \text{ order: } a_o \sim \frac{M}{r^2} \tag{4-12}$$

$$1^{st} \text{ order: } a_1 \sim \frac{m}{M} a_o \sim \frac{m}{r^2} \tag{4-13}$$

$$2^{nd} \text{ order: } a_2 \sim v^2 a_o \sim \frac{M}{r} a_o \sim \frac{Mv^2}{r^2} \sim \frac{M^2}{r^3} \tag{4-14}$$

$$3^{rd} \text{ order: } a_3 \sim \frac{m}{M} a_2 \sim \frac{mv^2}{r^2} \sim \frac{mM}{r^3} \sim \frac{Mvw}{r^2} \tag{4-15}$$

where

$M \overset{\text{def}}{=}$ mass of the sun

$m \overset{\text{def}}{=}$ mass of a planet

$r \overset{\text{def}}{=}$ typical interplanetary or planet-sun distance

$v \overset{\text{def}}{=}$ typical velocity of a planet relative to the sun

$w \overset{\text{def}}{=}$ hypothetical "absolute velocity of the sun" [cf. Eq. (4-7)] (4-16)

We will not bother going beyond the third order and the author even has some doubts about how seriously one should take the third order equations since $\vec{w}$ may appear in them. Of course, the $\vec{w}$ terms appear in third order only because we have guessed the size of $\vec{w}$ to be given by Eq. (4-7), and if this is wrong they could first appear in a different order. We will see that the first order equations are the usual planetary equations [Eqs. (4-3)], while the second order corrections give the terms which produce the relativistic advance of the perihelion of Mercury and the other planets. It follows that the influence of third order terms will probably be very hard to detect and fourth order terms will undoubtedly be irrelevant. If we wish to include third order terms we would have to assume a definite value for $\vec{w}$ and the only assumption which immediately suggests itself is De Sitter's,[47] viz., that the absolute velocity of the center of mass of the solar system vanishes. This determines $\vec{w}$, which then has the magnitude assumed in Eq. (4-7) and makes the third order terms well-determined. We shall write the third order equations including the explicit dependence on $\vec{w}$.

In obtaining the equations of our hierarchy it is helpful to note that if we define

$$\underset{\to n}{X} \overset{\text{def}}{=} X \text{ up to } n^{\text{th}} \text{ order} \tag{4-17}$$

then

$$\frac{d^2}{dt^2}\underset{\to n+1}{\vec{X}_{io}} = \underset{\to n+1}{\vec{F}_i} + \sum_{\text{all } j}\frac{m_j}{M}\underset{\to n}{\vec{F}_j} \tag{4-18}$$

and

$$\frac{d^2}{dt^2}\underset{\to n+1}{\vec{X}_{io}} - \frac{d^2}{dt^2}\underset{\to n}{\vec{X}_{io}} = \left(\underset{\to n+1}{\vec{F}_i} - \underset{\to n}{\vec{F}_i}\right) + \sum_{\text{all } j}\frac{m_j}{M}\left(\underset{\to n}{\vec{F}_j} - \underset{\to n-1}{\vec{F}_j}\right) . \tag{4-19}$$

It is now a straightforward task to identify the value of $\vec{F}_i$ up to a given order:

$$\underset{\to o}{\vec{F}_i} = -\frac{M\vec{X}_{io}}{r_{io}^3} \tag{4-20}$$

$$\underset{\to 1}{\vec{F}_i} = \underset{\to o}{\vec{F}_i} - \sum_{\text{all } j}\frac{m_j\vec{X}_{ij}}{r_{ij}^3} \tag{4-21}$$

$$\underset{\to 2}{\vec{F}_i} = \underset{\to 1}{\vec{F}_i} + \frac{4M^2\vec{X}_{io}}{r_{io}^4} + 4\vec{V}_{io}\frac{M}{r_{io}^3}\vec{X}_{io}\cdot\vec{V}_{io} - \frac{M\vec{X}_{io}}{r_{io}^3}V_{io}^2 \tag{4-22}$$

and

$$\underset{\to 3}{\vec{F}_i} = \underset{\to 2}{\vec{F}_i} + \frac{5m_i M\vec{X}_{io}}{r_{io}^4} + \frac{M}{2}\frac{\vec{X}_{io}}{r_{io}^3}\sum_j \frac{m_j}{r_{jo}^3}\vec{X}_{io}\cdot\vec{X}_{jo} + \frac{5M}{r_{io}^3}\vec{X}_{io}\sum_j\frac{m_j}{r_{ij}}$$

$$+ \frac{4M}{r_{io}}\sum_j \frac{m_j\vec{X}_{ij}}{r_{ij}^3} - \frac{M}{2}\sum_j m_j\frac{\vec{X}_{ij}}{r_{ij}^3}\frac{(\vec{X}_{ij}\cdot\vec{X}_{jo})}{r_{jo}^3} - \frac{7}{2}\frac{M}{r_{io}}\sum_j\frac{m_j\vec{X}_{jo}}{r_{jo}^3}$$

$$+ 4M\sum_j\frac{m_j\vec{X}_{ij}}{r_{ij}^3 r_{jo}} + 4\vec{V}_{io}\sum_j\frac{m_j}{r_{ij}^3}\vec{X}_{ij}\cdot\vec{X}_{io} - V_{io}^2\sum_j\frac{m_j\vec{X}_{ij}}{r_{ij}^3}$$

$$-2\sum_j m_j\frac{\vec{X}_{ij}}{r_{ij}^3}V_{jo}^2 + \frac{3}{2}\sum_j m_j\frac{\vec{X}_{ij}}{r_{ij}^5}(\vec{X}_{ij}\cdot\vec{V}_{ij})^2 - \frac{M}{2}\sum_j\frac{m_j}{r_{ij}^3 r_{jo}^3}\vec{X}_{ij}(X_{ij}\cdot\vec{X}_{jo})$$

$$-3\sum_j m_j\frac{\vec{V}_{ij}}{r_{ij}^3}\vec{X}_{ij}\cdot\vec{V}_{jo} + 4\sum_j m_j\frac{\vec{X}_{ij}}{r_{ij}^3}\vec{V}_{io}\cdot\vec{V}_{jo} - 4\sum_j m_j\frac{\vec{V}_{jo}}{r_{ij}^3}(\vec{V}_{io}\cdot\vec{X}_{ij})$$

$$+ \frac{3M}{r_{io}^3}\vec{X}_{io}(\vec{V}_{io}\cdot\vec{w}) + \frac{M}{r_{io}^3}\vec{V}_{io}(\vec{X}_{io}\cdot\vec{w}) \qquad \text{(sums exclude } j = i) \quad . \tag{4-23}$$

Using Eqs. (4-18) and (4-19) we find, in successive orders,

$$\frac{d^2}{dt^2}\underset{\to o}{\vec{X}_{io}} = -\frac{M\vec{X}_{io}}{r_{io}^3} \tag{4-24}$$

$$\frac{d^2}{dt^2}\underset{\to 1}{\vec{X}_{io}} = -(M + m_i)\frac{\vec{X}_{io}}{r_{io}^3} - \sum_{j\neq i} m_j\left(\frac{\vec{X}_{ij}}{r_{ij}^3} + \frac{\vec{X}_{jo}}{r_{jo}^3}\right) \tag{4-25}$$

$$\frac{d^2}{dt^2}\underset{\to 2}{\vec{X}_{io}} = -(M + m_i)\frac{\vec{X}_{io}}{r_{io}^3} - \sum_{j\neq i} m_j\left(\frac{\vec{X}_{ij}}{r_{ij}^3} + \frac{\vec{X}_{jo}}{r_{jo}^3}\right) - \frac{4M^2\vec{X}_{io}}{r_{io}^4}$$

$$+ \frac{4M}{r_{io}^3}\vec{V}_{io}(\vec{X}_{io}\cdot\vec{V}_{io}) - \frac{M}{r_{io}^3}\vec{X}_{io}V_{io}^2 \tag{4-26}$$

and

$$\frac{d^2}{dt^2}\underset{\to 3}{\vec{X}_{io}} = \frac{d^2}{dt^2}\underset{\to 2}{\vec{X}_{io}} + \left(\underset{\to 3}{\vec{F}_i} - \underset{\to 2}{\vec{F}_i}\right) + \sum_{\text{all } j}\frac{m_j}{M}\left(\underset{\to 2}{\vec{F}_i} - \underset{\to 1}{\vec{F}_i}\right) \quad . \tag{4-27}$$

These are our final equations. Except for an error in the coefficients in the third order acceleration, they were first found by Droste and De Sitter.[47]

V. GRT AND THE EARTH-MOON SYSTEM

Since the earth and moon are so close, we cannot use the order of magnitude decomposition of the last section for terms involving the earth-moon distance. On the other hand, we can certainly ignore third order terms arising from the sun and other planets in the equations of motion of the moon relative to the earth, and, if we desire, include them in the equations of motion of the earth-moon barycenter by merely using the sum of the masses of the earth and moon as the "mass" of the "barycenter." Since the "absolute velocity" of the sun does not appear, presumably, till the third order, we may as well choose our coordinate system centered in the sun.

We define

$$\vec{X}_M \overset{\text{def}}{=} \text{coordinates of moon relative to sun;}$$

$$r_M \overset{\text{def}}{=} |\vec{X}_M| \quad ; \quad \vec{V}_M \overset{\text{def}}{=} \frac{d\vec{X}_M}{dt} \tag{5-1}$$

$$\vec{X}_E \overset{\text{def}}{=} \text{coordinates of earth relative to sun;}$$

$$r_E \overset{\text{def}}{=} |\vec{X}_E| \quad ; \quad \vec{V}_E \overset{\text{def}}{=} \frac{d\vec{X}_E}{dt} \tag{5-2}$$

$$\vec{X}_{ME} \overset{\text{def}}{=} \vec{X}_M - \vec{X}_E \qquad \zeta \overset{\text{def}}{=} |\vec{X}_{ME}| \tag{5-3}$$

$$\frac{d^2\vec{X}}{dt^2} \overset{\text{def}}{=} \vec{a}^{(NR)} + \vec{a}^{(R)} \tag{5-4}$$

$$\left(\begin{Bmatrix} M \\ m \\ \mu \end{Bmatrix} \overset{\text{def}}{=} \text{mass of} \begin{Bmatrix} \text{sun} \\ \text{earth} \\ \text{moon} \end{Bmatrix}\right) \tag{5-5}$$

where $\vec{a}^{(NR)}$ is the nonrelativistic part and $\vec{a}^{(R)}$ the GRT part of the acceleration and only the sun, moon, and earth are considered in $\vec{a}^{(R)}$.

The post-Newtonian equations give us expressions for $\vec{a}_E^{(R)}$ and $\vec{a}_M^{(R)}$. Rather than writing these down, we write down the equivalent expressions for the relative earth-moon motion and the motion of the earth-moon barycenter:

$$\begin{aligned}
\frac{d^2\vec{X}_{ME}^{(R)}}{dt^2} = {} & \vec{X}_M \frac{M}{r_M^3} \left[\frac{4M + \mu}{r_M} + \left(4m + \frac{7}{2}\mu\right)\frac{1}{\zeta} + \frac{m}{r_E} - V_M^2\right] - \vec{X}_E \frac{M}{r_E^3} \Bigg[\frac{4M + m}{r_E} \\
& + \left(4\mu + \frac{7}{2}m\right)\frac{1}{\zeta} + \frac{\mu}{r_M} - V_E^2\Bigg] + \frac{\vec{X}_{ME}}{\zeta^3}\Bigg[\frac{2(2m + \mu)(2\mu + m)}{\zeta} + \frac{M(4m + \mu)}{r_M} \\
& + \frac{M(4\mu + m)}{r_E} - (2m + \mu) V_E^2 - (2\mu + m) V_M^2 + 4(m + \mu)\vec{V}_E \cdot \vec{V}_M \\
& + \frac{3}{2} m \frac{(\vec{V} \cdot \vec{X}_{ME})^2}{\zeta^2} + \frac{3}{2}\mu \frac{(\vec{V}_M \cdot \vec{X}_{ME})^2}{\zeta^2} - \frac{mM}{2r_E^3}\vec{X}_{ME} \cdot \vec{X}_{ES} \\
& + \frac{\mu M}{2r_M^3}\vec{X}_{ME} \cdot \vec{X}_{MS}\Bigg] + \vec{V}_M \Bigg[\frac{4M}{r_M^3}\vec{X}_{MS} \cdot \vec{V}_M + \frac{(4m + 3\mu)}{\zeta^3}\vec{X}_{ME} \cdot \vec{V}_M
\end{aligned}$$

$$- \frac{(3m + 4\mu)}{\xi^3} \vec{X}_{ME} \cdot \vec{V}_E \Big] - \vec{V}_E \left[\frac{4M}{r_E^3} \vec{X}_{ES} \cdot \vec{V}_E + \frac{(4m + 3\mu)}{\xi^3} \vec{X}_{ME} \cdot \vec{V}_M \right.$$

$$\left. - \frac{(3m + 4\mu)}{\xi^3} \vec{X}_{ME} \cdot \vec{V}_E \right] \quad . \tag{5-6}$$

For the earth-moon barycenter equations, let

$$m_T \overset{\text{def}}{=} m + \mu \tag{5-7}$$

and

$$m_T \vec{X}_B \overset{\text{def}}{=} m\vec{X}_E + \mu\vec{X}_M \quad .$$

Then

$$m_T \frac{d^2\vec{X}_B^{(R)}}{dt^2} = \vec{X}_M \frac{M\mu}{r_M^3} \left[\frac{4M + \mu}{r_M} + \frac{m}{r_E} + \frac{m}{2\xi} - V_M^2 \right] + \vec{X}_E \frac{Mm}{r_E^3} \left[\frac{4M + m}{r_E} \right.$$

$$\left. + \frac{\mu}{r_M} + \frac{\mu}{2\xi} - V_E^2 \right] + \vec{X}_{ME} \frac{\mu m}{\xi^3} \left[- \frac{(m - \mu)}{\xi} + \frac{4M}{r_M} - \frac{4M}{r_e} + V_M^2 - V_E^2 \right.$$

$$+ \frac{3(\vec{V}_E \cdot \vec{X}_{ME})^2}{\xi^2} - \frac{3(\vec{V}_M \cdot \vec{X}_{ME})^2}{\xi^2} - \frac{M}{r_E^3} \vec{X}_{ME} \cdot \vec{X}_{ES} - \frac{M}{r_M^3} \vec{X}_{ME} \cdot \vec{X}_{MS}$$

$$\left. + \frac{M}{r_E} - \frac{M}{r_M} \right] + \vec{V}_E m \left[-\mu \frac{\vec{X}_{ME} \cdot \vec{V}_E}{\xi^3} - \mu \frac{\vec{X}_{ME} \cdot \vec{V}_M}{\xi^3} + \frac{4M}{r_E^3} \vec{X}_{ES} \cdot \vec{V}_E \right]$$

$$+ \vec{V}_M \mu \left[\frac{m\vec{X}_{ME} \cdot \vec{V}_E}{\xi^3} + m \frac{\vec{X}_{ME} \cdot \vec{V}_M}{\xi^3} + \frac{4M}{r_M^3} \vec{X}_{MS} \cdot \vec{V}_M \right] \quad . \tag{5-8}$$

Essentially these equations were again first obtained and studied by De Sitter[47] who found a very interesting effect, essentially due to the velocity-dependence of the Schwarzschild force of the sun, which secularly adds an extra angular velocity to earth satellites, e.g., the moon. We will discuss this effect here because, on the basis of conversations with several relativists, the author feels that knowledge of this effect is not widespread. (The only text which discusses it is Eddington's. He calls it the De Sitter-Schouten Effect.[50]) To see the effect in its purest form, let us look at the terms in Eq. (4-6) which are due to the sun alone, i.e., which are present if we set the masses of the earth and of the moon equal to zero. Assume also that the earth moves in a circle about the sun whose radius is determined by Newton's Law for circular orbits:

$$\vec{X}_E = \vec{R} \tag{5-9}$$

$$\vec{X}_M = \vec{R} + \vec{\xi} \tag{5-10}$$

$$\vec{V}_E = \vec{\omega} \times \vec{R} \tag{5-11}$$

$$\vec{v}_E^{\,2} = \frac{M}{r} \quad . \tag{5-12}$$

Idealizing further, let us look for relativistic terms which survive when $\zeta \to 0$, i.e., we are looking for GRT effects on the motion of a very close satellite orbiting around a planet moving in a circle.

From Eq. (5-7) and Eqs. (5-9) through (5-11) we then find that the only terms that survive are

$$\frac{d^2\vec{\zeta}^{(R)}}{dt^2} \approx \frac{2M}{R^3}\left[2(\vec{\omega}\times\vec{R})\,(\vec{R}\cdot\frac{d\vec{\zeta}}{dt}) + \vec{R}(\vec{R}\cdot\vec{\omega}\times\frac{d\vec{\zeta}}{dt})\right] \quad . \tag{5-13}$$

Since we are looking for secular terms, we average this over the earth's motion. If the plane of the earth's orbit is called the x-y plane then obviously

$$\langle R_x^2\rangle = \langle R_y^2\rangle = \frac{1}{2}(\langle R_x^2\rangle + \langle R_y^2\rangle) = \frac{1}{2}\langle R^2\rangle = \frac{R^2}{2} \tag{5-14}$$

$$\langle R_xR_y\rangle = \langle R_xR_z\rangle = \langle R_yR_z\rangle = \langle R_z^2\rangle = 0 \tag{5-15}$$

or, in matrix notation,

$$\langle R_iR_j\rangle = \frac{1}{2}\,R^2\left(\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right)\ [\langle\vec{R}\,\vec{R}\rangle = \frac{1}{2}\,R^2(\underset{\approx}{1} - \hat{\omega}\hat{\omega})] \quad . \tag{5-16}$$

Thus we obtain

$$\left(\frac{d^2\vec{\zeta}}{dt^2}\right)^{(R)}_{\text{secular}} = \frac{3M}{R}\,\vec{\omega}\times\frac{d\vec{\zeta}}{dt} \quad . \tag{5-17}$$

Now, if the satellite equation ignoring relativity was

$$\frac{d^2\vec{\zeta}}{dt^2} = \vec{F} \tag{5-18}$$

then GRT changes it to

$$\frac{d^2\vec{\zeta}}{dt^2} - 2\vec{\Omega}\times\frac{d\vec{\zeta}}{dt} = \vec{F} \tag{5-19}$$

$$\vec{\Omega} \overset{\text{def}}{=} \frac{3M}{2R}\,\vec{\omega} \quad . \tag{5-20}$$

If we now transform Eq. (5-19) to a system of axes centered in the earth and rotating with angular velocity $\vec{\Omega}$, then, neglecting terms proportional to Ω^2, one obtains, in terms of the moving axes, just Eq. (5-18). So the De Sitter-Schouten Effect is that of a secular rotation of the nonrelativistic orbit of a satellite by an amount which depends only on the distance of the primary to the sun. For the case of the moon, this effect amounts to about 1.9 seconds of arc per century. It therefore may very well be observable.

VI. PROPAGATION OF LIGHT RAYS IN THE SOLAR SYSTEM

We will make the usual assumption that the trajectory of a light-ray is the same as that of a massless particle (photon) so that the equations of motion are the equations of a geodesic corresponding to a massless particle (null geodesic):[4]

$$\frac{d^2x^i}{d\sigma^2} + \Gamma^i_{k\ell}\frac{dx^k}{d\sigma}\frac{dx^\ell}{d\sigma} = 0 \quad ; \quad g_{ik}\frac{dx^i}{d\sigma}\frac{dx^k}{d\sigma} = 0 \quad . \qquad \text{[Eq. (2-17)]}$$

These equations will determine both the path and the parameter σ which is called a "distinguished parameter along the geodesic." Since the concept of a distinguished parameter is important in our discussion of the Doppler Effect (see Sec. VII), we will expand somewhat on the origin of the last equations and on geodesics in general.

A non-null geodesic can be defined as a curve along which

$$\delta \int_A^B ds = 0 \qquad \text{[Eq. (2-11)]}$$

where

$$(ds)^2 = -g_{ij}dx^i dx^j \quad . \qquad \text{[Eq. (2-12)]}$$

If we parameterize the curve by

$$x^i = x^i(\sigma) \qquad (6\text{-}1)$$

then we want to make

$$\delta \int_{\sigma_1}^{\sigma_2} \sqrt{-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}\, d\sigma = 0 \quad . \qquad (6\text{-}2)$$

The Euler-Lagrange equations for this problem lead to[51]

$$\frac{d^2x^i}{d\sigma^2} + \Gamma^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = \frac{d}{d\sigma}\left(\ln\frac{ds}{d\sigma}\right)\frac{dx^i}{d\sigma} \qquad (6\text{-}3)$$

assuming that

$$\frac{ds}{d\sigma} \neq 0 \qquad \text{(non-null condition)} \quad . \qquad (6\text{-}4)$$

Since the parameter σ is arbitrary, we can choose it in such a way as to simplify the equations. We choose it to be proportional to the arc length along the non-null geodesic by setting

$$g_{ij}\frac{dx^i}{d\sigma_D}\frac{dx^j}{d\sigma_D} = \pm a^2 \qquad (6\text{-}5)$$

where a is an arbitrary positive number, in which case Eqs. (6-3) reduce to

$$\frac{d^2x^i}{d\sigma_D^2} + \Gamma^i_{jk}\frac{dx^i}{d\sigma_D}\frac{dx^j}{d\sigma_D} = 0 \quad . \qquad (6\text{-}6)$$

Any σ_D which leads to Eq. (6-6) is called a distinguished or special parameter along the geodesic. We see immediately that if σ_{D1} is distinguished, so is

$$\sigma_{D2} = A\sigma_{D1} + B \tag{6-7}$$

where A and B are arbitrary constants. In particular, we can always choose σ_D so that

$$g_{ij}\frac{dx^i}{d\sigma_D}\frac{dx^j}{d\sigma_D} = \pm 1 \quad . \tag{6-8}$$

Except for the sign, this looks just like

$$ds^2 = -g_{ij}dx^i dx^j \quad . \qquad \text{[Eq. (2-9)]}$$

The two-fold sign appears in Eq. (6-8) but not in Eq. (2-9) because Eq. (2-9), though traditional, is ambiguous. The right-hand side of Eq. (2-9) can be either positive or negative whereas the arc-length is conventionally a real number. For a curve which represents the path of a nonzero mass particle, the right-hand side of Eq. (2-9) is positive so there is no problem there, and it is tacitly understood that in other cases an extra minus sign will be inserted when necessary[52] for consistency.

The above analysis breaks down for null curves, i.e., curves along which

$$ds^2 = -g_{ik}dx^i dx^k = 0 \tag{6-9}$$

since the right-hand side of Eq. (6-3) is then meaningless. In order to modify the formalism to handle this case, we replace the variational principle of Eq. (6-2) by

$$\delta\int g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}\,d\sigma = 0 \tag{6-10}$$

where σ is no longer completely arbitrary but is restricted by the subsidiary condition

$$g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} = \begin{Bmatrix} +1 \\ 0 \\ -1 \end{Bmatrix} \quad . \tag{6-11}$$

This variational principle leads to Eq. (6-6) and, because of Eq. (6-11), σ can be identified with the arc-length when the geodesics are non-null curves. These curves which satisfy Eq. (6-6) and which are null curves are called null geodesics. One can show that Eq. (6-6) implies that

$$\frac{d}{d\sigma}\left(g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}\right) = 0 \tag{6-12}$$

so that the expression in Eq. (6-11) has a constant value along a given geodesic. As long as this number is nonzero we can normalize its absolute value to unity and then σ becomes the arc-length along the curve. It also follows from Eq. (6-12) that a geodesic which is null anywhere is null everywhere. σ can no longer be identified with the arc-length along a null geodesic. We have thus derived Eq. (2-17). If we wish to use a nondistinguished parameter along a null geodesic then we replace Eq. (2-17) by the more general form

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = -\frac{d^2\lambda}{d\sigma^2}\Big/\left(\frac{d\lambda}{d\sigma}\right)^2\frac{dx^i}{d\lambda} \tag{6-13}$$

$$g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} = 0 \quad . \tag{6-14}$$

In particular, if we choose the time as the parameter, Eqs. (6-13) and (6-14) become

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = -\frac{d^2t}{d\sigma^2}\Big/\left(\frac{dt}{d\sigma}\right)^2\frac{dx^\alpha}{d\sigma} \quad ; \quad (\alpha = 1, 2, 3) \tag{6-15}$$

$$\Gamma^4_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = -\frac{d^2t}{d\sigma^2}\Big/\left(\frac{dt}{d\sigma}\right)^2 \tag{6-16}$$

$$g_{ik}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \quad . \tag{6-17}$$

It can be shown that Eq. (6-16) is redundant, i.e., if Eq. (6-15) and Eq. (6-17) are satisfied then Eq. (6-16) holds automatically. Thus we need only consider Eqs. (6-15) and (6-17).

The geodesic Eqs. (6-15) and (6-17) involve Γ^α_{ij} and g_{ij}. The g_{ij}, up to the post-Newtonian terms, are given in Eqs. (2-68) to (2-70). It will turn out that we can ignore the post-Newtonian terms as their effect is so small as to be undetectable at present and in the near future. Thus, we can write

$$g_{\alpha\beta} = (1-\psi)\,\delta_{\alpha\beta} \tag{6-18}$$

$$g_{\alpha 4} = 0 \tag{6-19}$$

$$g_{44} = -1-\psi \tag{6-20}$$

where we have used units such that $c = G = 1$ and ψ is twice the Newtonian potential

$$\psi(\vec{x}) \overset{\text{def}}{=} -2\sum_i \frac{m_i}{|\vec{x}-\vec{x}_i|} - \frac{2M}{|\vec{x}|} \tag{6-21}$$

and where we have taken the origin of our coordinate system in the sun since to this order the velocity of the sun can be ignored.

The lowest approximation is to ignore the gravitational potential altogether, i.e.,

$$g^{(o)}_{\alpha\beta} = +\delta_{\alpha\beta} \quad ; \quad g^{(o)}_{44} = -1 \quad ; \quad \text{other } g_{ij} = 0 \quad . \tag{6-22}$$

This causes all the Γ^i_{jk} to vanish and leads to

$$\frac{d^2x^\alpha}{dt^2} = -\frac{d^2t}{d\sigma^2}\left(\frac{dt}{d\sigma}\right)^2\frac{dx^\alpha}{dt} \tag{6-23}$$

$$\frac{dx^\alpha}{dt}\frac{dx^\alpha}{dt} = 0 \quad . \tag{6-24}$$

Differentiating Eq. (6-24) and combining with Eq. (6-23), we find

$$\frac{d^2t}{d\sigma^2} = 0 \tag{6-25}$$

and

$$\frac{d^2x^\alpha}{dt^2} = 0 \tag{6-26}$$

so in this approximation the distinguished parameters and geodesics are

$$\sigma = At + B \tag{6-27}$$

and

$$x^\alpha = D^\alpha t + E^\alpha \tag{6-28}$$

where A, B, D^α and E^α are constants. Light travels in a straight line and since in our units the speed of light far away from matter is unity, we write

$$\underset{o}{x}^\alpha(t) = x^\alpha(t_o) + (t - t_o)\, n^\alpha \tag{6-29}$$

where n^α are the components of a unit constant vector

$$n^\alpha n^\alpha = \hat{n} \cdot \hat{n} = 1 \tag{6-30}$$

as the first approximation to the path of a light ray which passes through the point with coordinates $x^\alpha(t_o)$ at time t_o and is traveling in the direction of the unit vector $\hat{n}$. We can ignore the distinction between covariant and contravariant indices on space components of vectors since the metric is Minkowskian to this order, i.e., is given by Eq. (6-22).

We now find the corrections to this trajectory in the next approximation

$$x^\alpha(t) = \underset{1}{x}^\alpha(t) + \underset{o}{x}^\alpha(t) \quad . \tag{6-31}$$

The Christoffel symbols, Γ^α_{jk}, are to first order in ψ,

$$\Gamma^1_{11} = -\Gamma^1_{22} = -\Gamma^1_{33} = -\Gamma^1_{44} = -\frac{1}{2}\frac{\partial\psi}{\partial x^1}$$

$$\Gamma^1_{12} = \frac{1}{2}\frac{\partial\psi}{\partial x^2} \quad ; \quad \Gamma^1_{13} = \frac{1}{2}\frac{\partial\psi}{\partial x^3} \quad ; \quad \Gamma^1_{14} = 0, \quad \text{etc.} \tag{6-32}$$

Using Eqs. (6-32) and (6-29) for $\underset{o}{x}(t)$, we can convert Eqs. (6-15) and (6-17) into

$$\frac{d^2}{dt^2}\underset{1}{x}^\alpha = 2\frac{\partial\psi}{\partial x^\beta} n^\beta x^\alpha - \frac{\partial\psi}{\partial x^\alpha} \tag{6-33}$$

and

$$n^\alpha \frac{d}{dt}\underset{1}{x}^\alpha = \psi \quad . \tag{6-34}$$

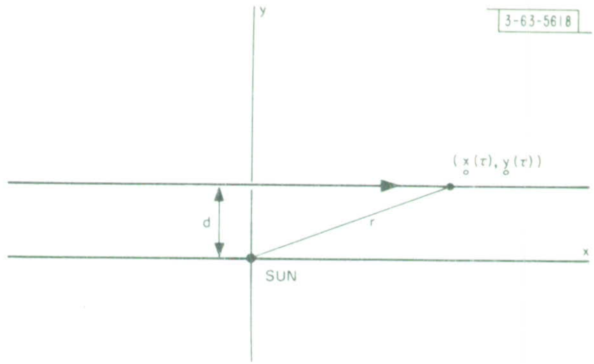
In vector notation, these become

$$\frac{d^2}{dt^2}\underset{1}{\vec{x}} = 2\hat{n}(\hat{n}\cdot\vec{\nabla})\,\psi - \vec{\nabla}\psi \tag{6-35}$$

and

$$\hat{n}\cdot\frac{d}{dt}\underset{1}{\vec{x}} = \psi \quad . \tag{6-36}$$

Fig. 1. Special coordinate system.



These equations can be integrated directly. However, to make things more transparent, let us look at a ray in the x-y plane which in lowest order moves in the positive x-direction. This obviously involves no loss of generality and, as we will see, our results can be transcribed easily into coordinate-free vectorial notation. Referring to Fig. 1, let the ray start out at $(x(\tau),\ y(\tau))$ at time τ with $\hat{n} = (1, 0)$. Then

$$x_o(t) = x_o(\tau) + (t - \tau_o) \qquad y_o(t) = y_o(\tau) \tag{6-37}$$

$$\dot{x}_o(t) = 1 \qquad \dot{y}_o(t) = 0 \quad . \tag{6-38}$$

The equations of motion for the next approximation become

$$\ddot{x}_1(t) = \frac{\partial\psi}{\partial x} \tag{6-39}$$

$$\ddot{y}_1(t) = -\frac{\partial\psi}{\partial y} \tag{6-40}$$

$$\dot{x}_1(t) = \psi \quad . \tag{6-41}$$

Integrating the equations

$$\dot{y}_1(t) = -\int_\tau^t \frac{\partial\psi(T)}{\partial y}\, dT + \dot{y}_1(\tau) \tag{6-42}$$

$$y_1(t) = -\int_\tau^t dT_1 \int_\tau^{T_1} dT\, \frac{\partial\psi(T)}{\partial y} + \dot{y}_1(\tau)\,(t-\tau) + y_1(\tau)$$

$$= -\int_\tau^t (t - T)\, \frac{\partial\psi(T)}{\partial y}\, dT + \dot{y}_1(\tau)\,(t-\tau) + y_1(\tau) \tag{6-43}$$

$$x_1(t) = \int_\tau^t \psi(T)\, dT + x_1(\tau) \tag{6-44}$$

where the integrals are evaluated along the zeroth-order trajectory, i.e., if

$$F = F(x, y, z, t)$$

$$\int F(T)\, dT \overset{\text{def}}{=} \int F\left[x_o(T),\ y_o(T),\ z_o(T),\ T\right] dT \quad . \tag{6-45}$$

Adding on the zeroth-order terms, we obtain the basic formulas

$$x(t) = x_0(\tau) + x_1(\tau) + (t - \tau) + \int_\tau^t \psi(T)\, dT \tag{6-46}$$

$$y(t) = y_0(\tau) + y_1(\tau) + \dot{y}_1(\tau)\,(t - \tau) - \int_\tau^t (t - T)\, \frac{\partial\psi}{\partial y}(T)\, dT \tag{6-47}$$

$$\dot{x}(t) = 1 + \psi(t) \tag{6-48}$$

$$\dot{y}(t) = \dot{y}_1(\tau) - \int_\tau^t \frac{\partial\psi}{\partial y}(T)\, dT \quad . \tag{6-49}$$

We now evaluate the integrals in the above equations for the part of ψ which is due to the sun since we will show later that this is all we need. There is no loss in generality in placing the sun at the origin of our coordinate system so that

$$\psi(x, y, z) = \frac{-2M}{\sqrt{x^2 + y^2 + z^2}} \tag{6-50}$$

and

$$\psi(T) = \frac{-2M}{\sqrt{[x_0(\tau) + (T - \tau)]^2 + [y_0(\tau)]^2}} \quad . \tag{6-51}$$

We find

$$\begin{aligned}\int_\tau^t \psi(T)\, dT &= -2M \int_\tau^t \frac{dT}{\{[x_0(\tau) + (T - \tau)]^2 + y_0^2(\tau)\}^{1/2}} \\ &= -2M \int_{x_0(\tau)}^{x_0(t)} \frac{d\alpha}{[\alpha^2 + y_0^2(\tau)]^{1/2}} \\ &= -2M \ln \left\{ \frac{[x_0^2(t) + y_0^2(\tau)]^{1/2} + x_0(t)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2} + x_0(\tau)} \right\}\end{aligned} \tag{6-52}$$

$$\begin{aligned}\int_\tau^t \frac{\partial\psi}{\partial y}\, dT = 2M\, y_0(\tau) \int_{x_0(\tau)}^{x_0(t)} \frac{d\alpha}{[\alpha^2 + y_0^2(\tau)]^{3/2}} &= \frac{2M}{y_0(\tau)} \left\{ \frac{x_0(t)}{[x_0^2(t) + y_0^2(\tau)]^{1/2}} \right. \\ &\quad \left. - \frac{x_0(\tau)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2}} \right\}\end{aligned} \tag{6-53}$$

$$\int_\tau^t dT_1 \int_T^{T_1} \frac{\partial\psi}{\partial y}(T)\, dT = \frac{2M}{y_0(\tau)} \left\{ [x_0^2(t) + y_0^2(\tau)]^{1/2} - [x_0^2(\tau) + y_0^2(\tau)]^{1/2} + \left[\frac{-(t - \tau)\, x_0(\tau)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2}} \right] \right\} \tag{6-54}$$

Therefore we finally obtain

$$x(t) = x_0(\tau) + (t-\tau) - 2M \ln\left\{\frac{[x_0^2(t) + y_0^2(\tau)]^{1/2} + x_0(t)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2} + x_0(\tau)}\right\} + x_1(\tau) \tag{6-55}$$

$$y(t) = y_0(\tau) + (t-\tau)\left\{\dot{y}_1(\tau) + \frac{2M}{y_0(\tau)} \frac{x_0(\tau)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2}}\right\} - \frac{2M}{y_0(\tau)}\{[x_0^2(t) + y_0^2(\tau)]^{1/2}$$

$$- [x_0^2(\tau) + y_0^2(\tau)]^{1/2}\} \tag{6-56}$$

$$\dot{x}(t) = 1 - \frac{2M}{[x_0^2(t) + y_0^2(\tau)]^{1/2}} \tag{6-57}$$

and

$$\dot{y}(t) + \frac{2M}{y_0(\tau)} \frac{x_0(t)}{[x_0^2(t) + y_0^2(\tau)]^{1/2}} = \dot{y}(\tau) + \frac{2M}{y_0(\tau)} \frac{x_0(\tau)}{[x_0^2(\tau) + y_0^2(\tau)]^{1/2}} \quad . \tag{6-58}$$

We can make the content of these formulas clearer if we observe from Fig. 1 that $y_0(\tau)$ is just "d," the distance of the closest approach of the undeflected ray to the sun, and that $[x_0^2(t) + y_0^2(\tau)]^{1/2}$ is just the distance of the point on the undeflected path from the sun:

$$x(t) = x_0(\tau) + (t-\tau) - 2M \ln \frac{r(t) + x_0(t)}{r(\tau) + x_0(\tau)} + x_1(\tau) \tag{6-59}$$

$$y(t) = d + (t-\tau)\left[\dot{y}_1(\tau) + \frac{2M}{d} \frac{x_0(\tau)}{r(\tau)}\right] - \frac{2M}{d}[r(t) - r(\tau)] \tag{6-60}$$

$$\dot{x}(t) = 1 - \frac{2M}{r(t)} \tag{6-61}$$

$$\dot{y}_1(t) + \frac{2M}{d} \frac{x_0(t)}{r(t)} = \dot{y}_1(\tau) + \frac{2M}{d} \frac{x_0(\tau)}{r(\tau)} \quad . \tag{6-62}$$

These formulas will now be applied to obtain the results of interest to us:

A. APPARENT SPEED AND ABERRATION OF LIGHT

For an observer at rest relative to the sun, the apparent speed of light $\bar{c}$ and the angle α which the ray will make with the x-axis are obviously determined by the equations

$$\bar{c} \cos\alpha = \dot{x} = 1 + \dot{x}_1 \tag{6-63}$$

$$\bar{c} \sin\alpha = \dot{y} = \dot{y}_1 \quad . \tag{6-64}$$

$$\therefore \quad \bar{c} = \sqrt{(1 + \dot{x}_1)^2 + \dot{y}_1^2} \approx 1 + \dot{x}_1 \tag{6-65}$$

$$\tan\alpha = \dot{y}_1/(1+\dot{x}_1) \approx \dot{y}_1 \approx \alpha \quad . \tag{6-66}$$

B. DEFLECTION OF STARLIGHT SEEN FROM THE EARTH

Suppose the light ray arrives at the earth at the time τ:

$$x_0(\tau) = x_E(\tau) \quad ; \quad x_1(\tau) = 0$$

$$y_0(\tau) = d \quad ; \quad y_1(\tau) = 0 \quad . \tag{6-67}$$

Furthermore, let the light appear to come along the x-axis, i.e.,

$$\dot{y}_1(\tau) = 0 \quad . \tag{6-68}$$

Then the actual velocity of the light at any time is, from Eqs. (6-61) and (6-62),

$$\dot{x}(t) = 1 - \frac{2M}{r(t)} \tag{6-69}$$

$$\dot{y}_1(t) + \frac{2M}{d}\frac{x_0(t)}{r(t)} = \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} \tag{6-70}$$

and, by Eq. (6-66),

$$\alpha(t) = \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} - \frac{2M}{d}\frac{x_0(t)}{r(t)} \tag{6-71}$$

As $t \to -\infty$, $x_0(t) \to -\infty$ and $r(t) \to |x_0(t)|$ so

$$\alpha(-\infty) = \lim_{t\to-\infty} \alpha(t) = \frac{2M}{d} + \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} \quad . \tag{6-72}$$

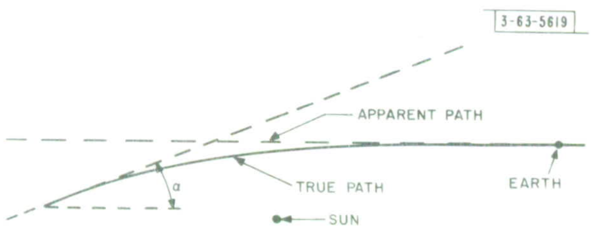


Fig. 2. Path of light ray.

We illustrate this result pictorially in Fig. 2 for the case where the earth and the star are on opposite sides of the sun. Note that the deflection increases monotonically along the path and that the deflection between $t = -\infty$ and the closest approach to the sun is $2M/d$. The ray must approach the sun closely for the bending to be significant since

$$\frac{M}{R_\odot} \sim 0''.45 \tag{6-73}$$

So if we assume $x_E \gg d$,

$$\alpha(-\infty) \approx \frac{4M}{d} = 1''.75 \left(\frac{R_\odot}{d}\right) \tag{6-74}$$

which is a classic result.[53]

C. ABERRATION OF STARLIGHT RELATIVE TO ABERRATION OF A PLANET SEEN FROM THE EARTH

Suppose a planet and a star appear to be along a straight line, i.e., their images coincide as seen from the earth. If the light rays have passed near the sun on their way to the earth then their directions have changed relative to what they would have been had they traveled in straight lines. We now calculate this shift.

Let us first assume that the earth is at rest relative to the sun and again let the light appear to come along the x-axis. Then we can use Eq. (6-72) for the starlight:

$$\alpha_{st} = \frac{2M}{d} + \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} \quad . \qquad \text{[Eq. (6-72)]}$$

Let the light leave the planet at time $\tau - \tau_1$. If it appears to come along the x-axis, then by Eqs. (6-61) and (6-62)

$$\dot{y}_1(\tau - \tau_1) + \frac{2M}{d}\frac{x_p(\tau - \tau_1)}{r_p(\tau - \tau_1)} = \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} \tag{6-75}$$

$$\alpha_p(\tau - \tau_1) = \frac{2M}{d}\frac{x_E(\tau)}{r_E(\tau)} - \frac{2M}{d}\frac{x_p(\tau - \tau_1)}{r_p(\tau - \tau_1)} + \frac{2M}{r_p(\tau - \tau_1)} \quad . \tag{6-76}$$

The last term can obviously be dropped since it is $\sim d/r_p$ as big as the other two and so combining Eq. (6-76) and Eq. (6-72) we obtain

$$\alpha_{st} - \alpha_p = \frac{2M}{d}\left[1 + \frac{x_p(\tau - \tau_1)}{r_p(\tau - \tau_1)}\right] \quad . \tag{6-77}$$

This is not a very practical form for the result, since we would like to express it directly in terms of the relative coordinates of the planet and of the earth rather than in terms of "x_p" and "d." Consider the straight line passing through the points $\vec{r}_1$ and $\vec{r}_2$:

$$\vec{r}(\lambda) = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1) \overset{\text{def}}{=} \vec{r}_1 + \lambda\vec{R} \qquad -\infty < \lambda < \infty \tag{6-78}$$

$$\vec{r}(0) = \vec{r}_1 \quad , \quad \vec{r}(1) = \vec{r}_2 \quad . \tag{6-79}$$

The distance $\Delta(\lambda)$ between the point $\vec{r}(\lambda)$ on the line and the sun (at the origin) is given by

$$\Delta^2(\lambda) = |\vec{r}_1 + \lambda\vec{R}|^2 = r_1^2 + 2\lambda\vec{R}\cdot\vec{r}_1 + \lambda^2R^2 \quad . \tag{6-80}$$

The distance of closest approach is obtained by differentiating Eq. (6-80):

$$0 = \frac{d\Delta^2}{d\lambda} = 2\vec{R}\cdot\vec{r}_1 + 2\lambda R^2 \quad .$$

$$\therefore \quad \lambda_{\text{at minimum}} = -\frac{\vec{R}\cdot\vec{r}_1}{R^2} \quad . \tag{6-81}$$

Now

$$d^2 = \Delta^2_{min} = \frac{1}{R^2}|(\vec{r}_1 \times \vec{r}_2)|^2 = \frac{r_1^2 r_2^2}{R^2}\sin^2 \sphericalangle(\vec{r}_1, \vec{r}_2) \tag{6-82}$$

and

$$\therefore \quad d = \frac{r_1 r_2}{R} \left| \sin \sphericalangle (\vec{r}_1, \vec{r}_2) \right| \tag{6-83}$$

where

$$R = |\vec{r}_1 - \vec{r}_2| \quad .$$

We now can determine the quantities "x" and "d" for a straight line passing through the positions of the earth at time τ and of the planet at time $\tau - \tau_1$ in terms of the coordinates of the earth and of the planet in the earth-sun-planet plane:

$$\vec{r}_E = (r_E \cos\Theta_E, r_E \sin\Theta_E) \quad ; \quad \vec{r}_p = (r_p \cos\Theta_p, r_p \sin\Theta_p) \tag{6-84}$$

$$R^2 = r_E^2 + r_p^2 - 2r_E r_p \cos(\Theta_E - \Theta_p) \tag{6-85}$$

$$d = \frac{r_E r_p}{R} \left| \sin(\Theta_E - \Theta_p) \right| \tag{6-86}$$

$$x_p = \vec{r}_p \cdot \frac{(\vec{r}_E - \vec{r}_p)}{|\vec{r}_E - \vec{r}_p|} = \frac{r_p}{R} [r_E \cos(\Theta_E - \Theta_p) - r_p] \tag{6-87}$$

and the angle α is measured counterclockwise from $\vec{R}$.

In the above analysis of the deflection of light, we have assumed the earth to be at rest relative to the sun. When we take the earth's motion into account, we find that from the viewpoint of an observer on the moving earth the direction a ray of light appears to be traveling differs from the direction that would be seen by a stationary observer at the same point in space. This effect is called aberration and is calculated by making a Lorentz transformation from the sun's rest frame to the instantaneous rest frame of the observation point on the earth.[54] Since the relative earth-sun speed is about 10^{-4}c, and angle of arrival measurements are limited in precision, it is sufficiently accurate to use the lowest order aberration correction, which may be expressed as follows:[54]

$$\vec{A}_e \rightarrow \vec{A}_e + (\vec{A}_e \cdot \hat{V}_E) \vec{V}_E \overset{\text{def}}{=} \vec{A}_o \tag{6-87a}$$

where $\vec{A}_e$ and $\vec{A}_o$ are vectors in the direction of the ray in the rest frames of the earth and sun respectively, and $\vec{V}_E$ is the velocity of the earth relative to the sun (the fact that $\vec{A}_e$ and $\vec{A}_o$ have different magnitudes is irrelevant since we are only interested in the directions along which they point). When Eq. (6-87a) is combined with the gravitational deflection we obtain

$$\vec{e}_o = \hat{e} + \frac{2M}{r_E} \frac{\hat{r}_E - (\hat{r}_E \cdot \hat{e})\hat{e}}{1 - \hat{r}_E \cdot \hat{e}} + (\hat{e} \cdot \hat{V}_E) \vec{V}_E \tag{6-87b}$$

$\hat{e} \overset{\text{def}}{=}$ unit vector along direction of ray in earth's rest system

$\vec{e}_o \overset{\text{def}}{=}$ vector along direction of ray in sun's rest system

$\vec{r}_E \overset{\text{def}}{=}$ vector from sun to earth .

We can rewrite Eq. (6-87b) for a geocentric coordinate system. Let

$$
\begin{aligned}
\hat{n} &\stackrel{\text{def}}{=} \text{unit vector \underline{from} earth \underline{to} point in sky from which light appears to be coming} \\
\vec{r}_s &\stackrel{\text{def}}{=} \text{vector \underline{from} earth \underline{to} sun} \\
\vec{V}_s &\stackrel{\text{def}}{=} \text{velocity of sun relative to earth} \quad .
\end{aligned}
\tag{6-87c}
$$

Then we make the following substitutions in Eq. (6-87b)

$$
\begin{aligned}
\hat{e} &\to -\hat{n} \\
\vec{r}_E &\to -\vec{r}_s \\
\vec{V}_E &\to -\vec{V}_s
\end{aligned}
\tag{6-87d}
$$

and arrive at the final formula

$$\vec{n}_o = \hat{n} + \frac{2M}{r_s} \frac{\hat{r}_s - (\hat{r}_s \cdot \hat{n})\hat{n}}{1 - \hat{r}_s \cdot \hat{n}} \mp (\hat{n} \cdot \hat{V}_s)\vec{V}_s \tag{6-87e}$$

D. TIME OF FLIGHT OF LIGHT

Suppose we have two bodies, say the earth and a planet, whose trajectories $\vec{r}_E(t)$ and $\vec{r}_p(t)$ are assumed known. We send light (or a radar beam) from $\vec{r}_E$ at a time t_1 and want to know at what time $t_1 + T$ it will reach the planet. If GRT could be ignored, we would aim the beam along the straight line from $\vec{r}_E(t_1)$ to $\vec{r}_p(t_1 + T)$ and the light would travel along this line with a speed of unity (in our units). Thus the trajectory of the beam in flat space would be

$$\vec{r}(t) = \vec{r}_E(t_1) + \frac{\vec{r}_p(t_1 + T) - \vec{r}_E(t_1)}{|\vec{r}_p(t_1 + T) - \vec{r}_E(t)|}(t - t_1) \qquad \text{(flat space)} \tag{6-88}$$

and the time of flight, T, must be obtained from the implicit equation

$$\vec{r}_p(t_1 + T) = \vec{r}_E(t_1) + \frac{\vec{r}_p(t_1 + T) - \vec{r}_E(t_1)}{|\vec{r}_p(t_1 + T) - \vec{r}_E(t_1)|}\, T \tag{6-89}$$

or

$$T = |\vec{r}_p(t_1 + T) - \vec{r}_E(t_1)| \qquad \text{(flat space)} \quad . \tag{6-90}$$

GRT introduces two corrections: The actual path is not a straight line and the local speed of light is not unity. This can be seen from Eqs. (6-59) to (6-62) which automatically take these effects into account. Choosing the x-axis to connect $\vec{r}_E(t_1)$ to $\vec{r}_p(t_1 + T)$,

$$x(t) = x_E(t_1) + (t - t_1) - 2M \ln\left[\frac{r(t) + x(t)}{r_E(t_1) + x_E(t_1)}\right] \tag{6-91}$$

and the time of flight, T, is to be determined from

$$x_p(t_1 + T) = x_E(t_1) + T - 2M \ln\left[\frac{r_p(t_1 + T) + x_p(t_1 + T)}{r_E(t_1) + x_E(t_1)}\right] \quad . \tag{6-92}$$

The last equation can be rewritten in terms of

$$\vec{R}(t_a, t_b) \overset{\text{def}}{=} \vec{r}_p(t_a) - \vec{r}_E(t_b) \tag{6-93}$$

$$\hat{n}(t_a, t_b) \overset{\text{def}}{=} \vec{R}/|\vec{R}| \tag{6-94}$$

$$x_p(t_1 + T) = \frac{\vec{R}(t_1 + T, t_1)}{R} \cdot \vec{r}_p(t_1 + T) \tag{6-95}$$

$$x_E(t_1) = \frac{\vec{R}(t_1 + T, t_1)}{R} \cdot \vec{r}_E(t_1) \tag{6-96}$$

as (Ref. 65)

$$T = R(t_1 + T, t_1) + 2M \ln\left[\frac{r_p(t_1 + T) + \hat{n} \cdot \vec{r}_p(t_1 + T)}{r_E(t_1) + \hat{n} \cdot \vec{r}_E(t_1)}\right] \tag{6-97}$$

which is to be contrasted with the flat-space equation (6-90).

If the signal is reflected back from the planet so that it reaches the earth at time $t_1 + T + T_1$, then Eq. (6-59) gives for the return trip

$$x(t) = x_p(t_1 + T) + (t - T - t_1) - 2M \ln\left[\frac{r(t) + x(t)}{r_p(t_1 + T) + x_p(t_1 + T)}\right] \tag{6-98}$$

where the positive x-axis is now in the direction of $\vec{r}_E(t_1 + T + T_1) - \vec{r}_p(t_1 + T)$, i.e., essentially reversed:

$$x(t) = \frac{\vec{r}_E(t_1 + T + T_1) - \vec{r}_p(t_1 + T)}{|\vec{r}_E(t_1 + T + T_1) - \vec{r}_p(t_1 + T)|} \cdot \vec{r}(t) = -\hat{n}(t_1 + T,\ t_1 + T + T_1) \cdot \vec{r}(t) \tag{6-99}$$

so

$$T_1 = R(t_1 + T,\ t_1 + T + T_1) + 2M \ln\left[\frac{r_E(t_1 + T + T_1) - \hat{n}' \cdot \vec{r}_E(t_1 + T + T_1)}{r_p(t_1 + T) - \hat{n}' \cdot \vec{r}_p(t_1 + T)}\right] \tag{6-100}$$

$$\hat{n}' \overset{\text{def}}{=} \hat{n}(t_1 + T,\ t_1 + T + T_1) \quad . \tag{6-101}$$

Note that the arguments of the logarithms in Eqs. (6-97) and (6-100) are essentially equal as a consequence of the identities

$$(r_p + \hat{n} \cdot \vec{r}_p)(r_p - \hat{n} \cdot \vec{r}_p) = r_p^2 - (\hat{n} \cdot \vec{r}_p)^2 = (\vec{r}_p \times \hat{n}) \cdot (\vec{r}_p \times \hat{n})$$

$$= \frac{(\vec{r}_p \times \vec{r}_E)^2}{R^2} = d^2$$

and

$$(r_E + \hat{n} \cdot \vec{r}_E)(r_E - \hat{n} \cdot \vec{r}_E) = r_E^2 - (\hat{n} \cdot r_E)^2 = (\vec{r}_E \times \hat{n}) \cdot (\vec{r}_E \times \hat{n})$$

$$= \frac{(\vec{r}_p \times \vec{r}_E)^2}{R^2} = d^2 \quad . \tag{6-102}$$

The GRT one-way time delay is thus approximately doubled for a round trip. In terms of the distance of closest approach to the sun, the outbound GRT delay is

$$(\delta t)_{out} = 2M \ln \left[\frac{r_p \pm \sqrt{r_p^2 - d^2}}{r_E - \sqrt{r_E^2 - d^2}}\right] \quad . \tag{6-103}$$

Use of plus or minus is determined by whether the earth and planet are on the opposite or same side of the sun, respectively. This is obviously greatest near superior conjunction and then we can write

$$(\delta t)_{out} \approx 2M \ln \left(\frac{4r_p r_E}{d^2}\right) \quad . \tag{6-104}$$

For $d = R_o$, this term is about 10^{-4} sec for Venus and Mercury. We can also rewrite $(\delta t)_{out}$ in terms of the earth-planet coordinates in the earth-planet-sun plane as given in Eqs. (6-84) to (6-86):

$$(\delta t)_{out} = 2M \ln \left[\frac{r_E + r_p + R}{r_E + r_p - R}\right]$$

$$R = [r_E^2 + r_p^2 - 2r_E r_p \cos(\Theta_E - \Theta_p)]^{1/2} \quad . \tag{6-105}$$

The expression for the return trip has exactly the same form but the various quantities are evaluated at different times. For later use, we also calculate:

$$\frac{d}{dt}(\delta t)_{out} = 2M \left[\frac{\dot{r}_E + \dot{r}_p + \dot{R}}{r_E + r_p + R} - \frac{\dot{r}_E + \dot{r}_p - \dot{R}}{r_E + r_p - R}\right] \tag{6-106a}$$

which for the case of coplanar orbits becomes

$$\frac{d}{dt}(\delta t)_{out} = \frac{2M}{R}(\dot{r}_E r_p - \dot{r}_p r_E)\left(\frac{r_E - r_p}{r_p r_E}\right) + \frac{2M}{R}(r_E + r_p)(\dot{\Theta}_E - \dot{\Theta}_p)\tan[\tfrac{1}{2}(\Theta_E - \Theta_p)] \quad . \tag{6-106b}$$

The first term is always small ($\lesssim 10^{-12}$) but the second term becomes much larger near superior conjunction. We then have

$$\tan[\tfrac{1}{2}(\Theta_E - \Theta_p)] \approx \pm \frac{2r_E r_p}{(r_E + r_p)\, d} \quad ; \quad (\pm) \text{ if } \Theta_E - \Theta_p \binom{<}{>} \pi \tag{6-107}$$

and

$$\frac{d}{dt}(\delta t)_{out} \approx \pm 4M \frac{r_E r_p}{r_E + r_p} \frac{\dot{\Theta}_E - \dot{\Theta}_p}{d} \tag{6-108}$$

which has the values, for coplanar orbits

$$\frac{d}{dt}(\delta t)_{out} \approx \mp 2.1 \times 10^{-10}\left(\frac{R_o}{d}\right) \qquad \text{Venus}$$

$$\mp 6.9 \times 10^{-10}\left(\frac{R_o}{d}\right) \qquad \text{Mercury} \tag{6-109}$$

The basic equations (6-97) and (6-98) are solved by an iteration scheme using the best known values of $\vec{r}_E(t)$ and $\vec{r}_p(t)$.[55] Setting $t_1 = 0$, which entails no loss of generality, the equations to solve are

$$\left.\begin{aligned} T &= F(T, 0) \\ T' &= F(T, T') \end{aligned}\right\} \tag{6-110}$$

$$\left.\begin{aligned} F(t_A, t_B) &= R(t_A, t_B) + \Delta(t_A, t_B) \\ R(t_A, t_B) &= |\vec{r}_p(t_A) - \vec{r}_E(t_B)| \\ \Delta(t_A, t_B) &= 2M \ln \left\{\frac{r_E(t_B) + r_p(t_A) + R(t_A, t_B)}{r_E(t_B) + r_p(t_A) - R(t_A, t_B)}\right\} \end{aligned}\right\} \quad . \tag{6-111}$$

We then iterate

$$\left.\begin{aligned} T_1 &= F(0, 0) \\ T_2 &= F(T_1, 0) \\ T_3 &= F(T_2, 0) \end{aligned}\right\} \tag{6-112}$$

$$\left.\begin{aligned} T_1' &= F(T_3, T_3) \\ T_2' &= F(T_3, T_3 + T_1') \\ T_3' &= F(T_3, T_3 + T_2') \end{aligned}\right\} \quad . \tag{6-113}$$

To show that this is sufficiently accurate, let us look at Eq. (6-112) and Eq. (6-110). Then

$$\begin{aligned} T - T_{N+1} &= F(T, 0) - F(T_N, 0) \\ &= \left.\frac{\partial F}{\partial t_A}\right|_{(t_A = T_N + \alpha(T - T_N),\, t_B = 0)} (T - T_N) \overset{\text{def}}{=} (T - T_N)\,\sigma \end{aligned} \tag{6-114}$$

where $0 \leqslant \alpha \leqslant 1$ and where we have used the mean value theorem of elementary calculus. This can be rewritten as

$$T - T_{N+1} = \frac{\sigma(T_{N+1} - T_N)}{1 - \sigma} \quad . \tag{6-115}$$

By repeatedly applying the mean value theorem and using the fact that the time derivative has the same order of magnitude throughout, we conclude that

$$T - T_{N+1} \approx \frac{\sigma^{N+1}}{1 - \sigma} T_1 \quad . \tag{6-116}$$

Now, $T_1 \lesssim 1000$ sec and

$$\frac{dR}{dt} = (\vec{V}_p - \vec{V}_E) \cdot \frac{\vec{R}}{R} \tag{6-117}$$

so if we combine Eq. (6-117) and Eq. (6-106) we find $\sigma \leqslant 2 \times 10^{-4}$ and $|T - T_3| \leqslant 4 \times 10^{-9}$ sec. The time of flight is thus given by $T_3 + T_3'$ to within 10^{-8} sec. In view of Eq. (6-106) it is not really necessary to iterate Δ more than once; $\Delta(T_1, 0)$ can be used throughout.

We have ignored the effects of the gravitational fields of the planets on the time-delay. Let us show that these terms are in fact negligible. Going back to Eq. (6-46) and noting that the closest we come to the center of a planet is one planetary radius away, we see that

$$(\delta t)_{\substack{\text{planet-earth} \\ \text{round trip}}} \approx 4M_p \ln \frac{|\vec{r}_E - \vec{r}_p|}{R_p} + 4M_E \ln \frac{|\vec{r}_E - \vec{r}_p|}{R_E} \tag{6-118}$$

which is a time delay of less than 10^{-9} seconds for Venus and Mercury.

Finally, we wish to relate the value of the time-of-flight we have obtained to that which would be measured by an observer on the earth in terms of an atomic clock at the radar site. The invariant interval between two events is given by Eq. (2-9):

$$ds^2 = -g_{ij} dx^i dx^j \qquad \text{[Eq. (2-9)]}$$

which, in our approximation, becomes [cf. Eqs. (6-18) to (6-21)]:

$$\begin{aligned} ds^2 &= (1 + \psi)\, dt^2 - (1 - \psi)\,(dx^2 + dy^2 + dz^2) \\ &= dt^2 \left\{1 + \psi - (1 - \psi)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\} \\ &\approx dt^2 \{1 + \psi - v^2\} \overset{\text{def}}{=} d\tau^2 \quad . \end{aligned} \tag{6-119}$$

A fundamental postulate of GRT identifies this quantity with the square of the proper-time interval, i.e., the time measured by a clock at rest relative to the moving body. Put another way, if a clock at rest and very far away from all other masses would read "dt" then an identical clock on the earth would read "dτ."[56] The atomic clock on the earth then would measure a "proper-time-of-flight" of

$$\tau = \int_{t_1}^{t_1+T_f} [1 + \psi - v^2]^{1/2}\, dt \tag{6-120}$$

if t_1 is the coordinate time at which the signal leaves the earth.

The integrand in Eq. (6-120) is not constant due to the motion of the various bodies in the solar system and to the rotation of the radar site with the earth, so the relations between the coordinate-time and the local atomic-time epochs are quite complicated. Since M. Ash and I. Shapiro have studied this point extensively,[57] we will not discuss it here but will merely show that the variation of the integrand during the time-of-flight can be ignored, i.e., we can use

$$\tau = \{1 + \psi(t_1) - [v(t_1)]^2\}^{1/2}\, T_f \quad . \tag{6-121}$$

The next term in a Taylor Series expansion of τ would be

$$\frac{T_f^2}{2} \; \frac{\frac{1}{2}\frac{d\psi}{dt} - v\,\frac{dv}{dt}}{[1 + \psi - v^2]^{1/2}} \quad .$$

Now,

$$\vec{r} = \vec{r}_c + \vec{\rho}$$

$$\vec{v} = \vec{v}_c + \vec{\omega} \times \vec{\rho}$$

$$\vec{a} = \vec{a}_c + \vec{\omega} \times (\vec{\omega} \times \vec{\rho}) \tag{6-122}$$

where the subscript "c" refers to the center of the earth, $\vec{\omega}$ is the angular velocity of the earth's rotation, and $\vec{\rho}$ is a vector from the center of the earth to the radar site. We have

$$\frac{v_c}{c} \sim 10^{-4} \quad ; \quad \frac{\omega\rho}{c} \sim 10^{-6} \qquad \therefore \quad \frac{v}{c} \sim 10^{-4}$$

$$\frac{a_c}{c} \sim 1.6 \times 10^{-11} \quad ; \quad \frac{\omega^2\rho}{c} \sim 7 \times 10^{-11} \qquad \therefore \quad \frac{a}{c} \sim 10^{-10}$$

$$\therefore \left| T_f^2 \frac{v}{c^2} \frac{dv}{dt} \right| \lesssim 10^{-8} \text{ sec} \tag{6-123}$$

while if we just include the earth, moon, and sun in the gravitational potential,

$$T_f^2 \frac{d\psi}{dt} = \left(\frac{M_s}{r_s^2} \frac{dr_s}{dt} + \frac{M_m}{r_m^2} \frac{dr_m}{dt} \right) T_f^2 \tag{6-124}$$

where r_m and r_s are the distances to the moon and sun, respectively. These terms do not exceed 10^{-9} seconds. Thus we can use Eq. (6-121) for this experiment.

VII. THE DOPPLER EFFECT

We will begin by outlining the theory of geometrical optics in the Special Theory of Relativity.[58] We look for solutions of Maxwell's equations where the quantities describing the field, say the potentials, are of the form

$$A_\mu(x) = \mathrm{Re}\ [a_\mu(x)\ e^{i\psi(x)}] \tag{7-1}$$

where $\psi(x)$ is called the eikonal. This is the generalization of a plane wave for which

$$A_i(x) = \mathrm{Re}\ \{[a_i \exp[i(k_j x^j + \alpha)]\} = \mathrm{Re}\ \{[a_i \exp[i(\vec{k}\cdot\vec{r} - \omega t + \alpha)]\} \quad . \tag{7-2}$$

We want solutions of the form of Eq. (7-1) which are approximately plane waves, i.e., the amplitude and direction of the wave remain practically constant over distances of the order of a wavelength. Since $\psi(x)$ changes by 2π in one wavelength, $\psi(x)$ will be a large number for a macroscopic distance and furthermore, derivatives of ψ will be much bigger in size than derivatives of $a_i(x)$. Substituting Eq. (7-1) into Maxwell's equations, and making use of the assumed large value of $\psi(x)$, we obtain the eikonal equation:

$$\eta^{ij} \frac{\partial\psi}{\partial x^i} \frac{\partial\psi}{\partial x^j} = 0 \tag{7-3}$$

which we rewrite in terms of the wave-vector:

$$k_i \overset{\text{def}}{=} \frac{\partial\psi}{\partial x^i} \quad ; \quad k^i = (\vec{k}, \omega) \quad ; \quad \eta_{ij} k^i k^j = 0 \quad . \tag{7-4}$$

A surface of constant ψ is called a wave-surface and the wave-vector is normal to the surface, with its direction defining that of the ray (a curve whose tangent at each point coincides with the direction of propagation of the wave). Equation (7-3) looks like the Hamilton-Jacobi equation for a free particle[59]

$$\eta^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} + m^2 = 0 \tag{7-5}$$

for which the momentum and energy of the particle are given by

$$p_i = \frac{\partial S}{\partial x^i}$$

$$p^i = (\vec{P}, E) \quad ; \quad \eta_{ij} p^i p^j = -m^2 \tag{7-6}$$

so that a ray looks like the trajectory of a massless particle. The analogy can be made even stronger if we observe that Hamilton's equations

$$\frac{d\vec{P}}{dt} = -\vec{\nabla}E \quad ; \quad \vec{V} = \frac{d\vec{r}}{dt} = \frac{\partial E}{\partial \vec{p}} \tag{7-7}$$

have as analogues

$$\frac{d\vec{k}}{dt} = -\vec{\nabla}\omega \quad ; \quad \frac{d\vec{r}}{dt} = \frac{\partial\omega}{\partial\vec{k}} \tag{7-8}$$

so that in a vacuum where $|\vec{k}| = \omega$ the speed of light along the ray is unity and the rays are straight lines.

Now, by its definition, the ray is perpendicular to the wave surface, i.e., if dx^i represent an infinitesimal vector connecting two points on the wave surface then

$$0 = d\psi = \frac{\partial\psi}{\partial x^i}\, dx^i \quad . \tag{7-9}$$

But the ray also has the peculiar property of lying in the wave surface, i.e., if we consider an infinitesimal vector along the ray (ϵk^i), then it connects points on the wave surface. To prove this, note that

$$d\psi = \frac{\partial\psi}{\partial x^i}\, \epsilon k^i = \epsilon\eta^{ij} \frac{\partial\psi}{\partial x^j} = 0 \quad . \tag{7-10}$$

Let us now look at an emitter and receiver of light and consider two rays joining them.[60] In Fig. 3 we have drawn a space-time picture of the process. Note that ψ has the same value at each point along the ray since the ray lies in the wave surface. We now define the "unit" tangent vectors to the world lines of the emitter and receiver

$$u^i_{(1)} = \frac{dx^i_{(1)}}{ds_{(1)}} \quad ; \quad u^i_{(2)} = \frac{dx^i_{(2)}}{ds_{(2)}} \tag{7-11}$$

$$ds^2_{(1)} = -\eta_{ij} dx^i_{(1)} dx^j_{(1)} \quad ; \quad ds^2_{(2)} = -\eta_{ij} dx^i_{(2)} dx^j_{(2)} \tag{7-12}$$

$$u^i_{(1)} u^j_{(1)} \eta_{ij} = -1 = u^i_{(2)} u^j_{(2)} \eta_{ij}$$

and note that we can then compute $\Delta\psi$ in two ways:

$$\left(\frac{\partial\psi}{\partial x^i}\right)_{(1)} u^i_{(1)} \Delta s_{(1)} = \Delta\psi = \left(\frac{\partial\psi}{\partial x^i}\right)_{(2)} u^i_{(2)} \Delta s_{(2)} \tag{7-13}$$

or

$$k^i_{(1)} u_{(1)i} \Delta s_{(1)} = k^i_{(2)} u_{(2)i} \Delta s_{(2)} \tag{7-14}$$

or

$$\frac{\Delta s_1}{\Delta s_2} = \frac{(k^i u_i)_{(2)}}{(k^i u_i)_{(1)}} \quad . \tag{7-15}$$

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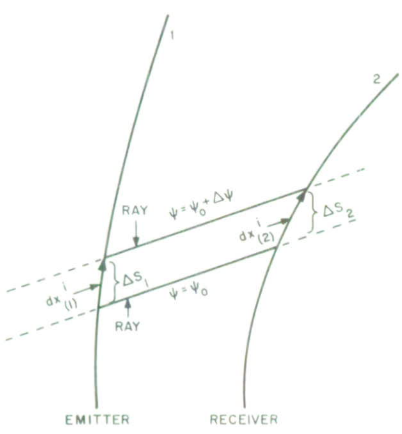

Fig. 3. Space-time diagram for Doppler effect and red shift.

But Δs is, by definition, just the proper-time interval, i.e., the time interval measured by a clock carried by the observer, and $\Delta\psi$ is just 2π times the number of waves each observer detects in this time interval and is the same for both. Thus the ratio of the frequencies as measured by the receiver and the transmitter is

$$\frac{f_2}{f_1} = \frac{\frac{\Delta\psi}{2\pi}/\Delta s_2}{\frac{\Delta\psi}{2\pi}/\Delta s_1} = \frac{\Delta s_1}{\Delta s_2} = \frac{(k^i u_i)_{(2)}}{(k^i u_i)_{(1)}} \quad . \tag{7-16}$$

This can be rewritten in a more familiar form if we note that the ratio in Eq. (7-16) is written in terms of Lorentz-invariant quantities and so can be evaluated in any coordinate system. Choose a system in which the transmitter is at rest. Then

$$dx^{\alpha}_{(1)} = 0 \quad ; \quad \frac{dx^4_{(1)}}{ds_{(1)}} = 1 = \frac{dt}{ds_{(1)}} \quad ; \quad k_i u^i_{(1)} = k_4$$

$$k_i u^i_{(2)} = k_i \frac{dx^i_{(2)}}{dt} \frac{dt}{ds_{(2)}} = (\vec{k} \cdot \vec{v}_{(2)} + k_4) \frac{1}{[1 - v^2_{(2)}]^{1/2}}$$

$$k_i k^i = 0 = |\vec{k}|^2 - k_4^2 \ .$$

$$\therefore \ k_4 = \eta_{44} k^4 = -\omega = -|\vec{k}| \quad ; \quad \therefore \ \frac{f_2}{f_1} = \frac{1 - \hat{n} \cdot \vec{v}_{(2)}}{[1 - v^2_{(2)}]^{1/2}} \quad . \tag{7-17}$$

The above procedure may seem like a rather long-winded way to obtain Eq. (7-17) but it has the advantage of being usable almost word for word in GRT.[58] The GRT generalization of Maxwell's equations in free space also admits an eikonal-type solution, only now

$$g^{ij} \frac{\partial\psi}{\partial x^i} \frac{\partial\psi}{\partial x^j} = 0 \tag{7-18}$$

which we can again rewrite as

$$k_i \stackrel{\text{def}}{=} \frac{\partial\psi}{\partial x^i} \quad , \quad g^{ij} k_i k_j = 0 \quad . \tag{7-19}$$

Proceeding exactly as before, but with g_{ij} replacing η_{ij}, we again end up with the ratio of proper time frequencies

$$\frac{f_2}{f_1} = \frac{(g_{ij} k^i u^j)_{(2)}}{(g_{ij} k^i u^j)_{(1)}} \quad . \tag{7-20}$$

Before we can use this formula, however, we need the relationship between $k^i_{(1)}$ and $k^i_{(2)}$. In the special theory, these quantities are equal and the rays are null geodesics, i.e., straight lines whose tangents have zero length. This suggests the GRT result.[61] If we define a curve by means of

$$\frac{dx^i}{d\lambda} = k^i = g^{ij} \frac{\partial\psi}{\partial x^j} \tag{7-21}$$

then a straightforward computation shows that because of Eq. (7-19)

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

$$g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \tag{7-22}$$

so that if we know the equation of the null geodesic connecting the transmitter to the receiver and the special parameter along it, we can relate $k^i_{(1)}$ and $k^i_{(2)}$ by means of Eq. (7-21):

$$\frac{f_2}{f_1} = \left(g_{ij}\frac{dx^i}{d\lambda}u^j\right)_{(2)} \Big/ \left(g_{ij}\frac{dx^i}{d\lambda}u^j\right)_{(1)} \quad . \tag{7-23}$$

Fortunately for us, the special parameter is easy to find when the metric is "static," i.e., there exists a coordinate system in which the g_{ij} do not depend on time and in which $g_{14} = g_{24} = g_{34} = 0$ (Ref. 62). This can be seen from the variational principle in Eq. (6-10) for the static case

$$\delta\int\left[g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} + g_{44}\frac{dt}{d\sigma}\frac{dt}{d\sigma}\right]d\sigma = 0 \quad . \tag{7-24}$$

The Euler-Lagrange equation for δt immediately leads to

$$\frac{d}{d\sigma}\left(g_{44}\frac{dt}{d\sigma}\right) = 0 \qquad \text{or} \qquad d\sigma = (-g_{44})\, dt \tag{7-25}$$

since changing σ by a constant factor changes nothing.

We can then finally write

$$\frac{f_2}{f_1} = \left[g_{ij}\frac{1}{-g_{44}}\frac{dx^i}{dt}\frac{dx^j_{(2)}}{ds_{(2)}}\right]_{(2)} \Big/ \left[g_{ij}\frac{1}{-g_{44}}\frac{dx^i}{dt}\frac{dx^j_{(1)}}{ds_{(1)}}\right]_{(1)} \quad . \tag{7-26}$$

This formula obviously reduces to the special relativistic one when $g_{ij} = \eta_{ij}$. Suppose now that both observers are at rest. Then

$$dx^1_{\substack{(1)\\(2)}} = dx^2_{\substack{(1)\\(2)}} = dx^3_{\substack{(1)\\(2)}} = 0 \quad ; \qquad ds_{\substack{(1)\\(2)}} = \left[-g_{44\substack{(1)\\(2)}}\right]^{1/2} dt$$

$$\frac{f_2}{f_1} = \left[g_{44}\frac{1}{-g_{44}}(-g_{44})^{-1/2}\right]_{(2)} \Big/ \left[g_{44}\frac{1}{-g_{44}}(-g_{44})^{-1/2}\right]_{(1)}$$

$$= \left(\frac{-g_{44(1)}}{-g_{44(2)}}\right)^{1/2} \tag{7-27}$$

which is just the usual red-shift formula. GRT thus gives us a combined "Red-Doppler" frequency shift.

We now turn to the implications of Eq. (7-26) for an interplanetary radar experiment. We send out a signal at frequency f_1, it is received and reflected by the planet at frequency f_2, and returns to the earth with frequency f_3. We are interested in

$$f_3 - f_1 = f_1\left(\frac{f_3}{f_1} - 1\right) \tag{7-28}$$

and we want $f_3 - f_1$ to one cps if $f_1 \sim 10^{10}$ cps, so we need the ratio up to one part in 10^{-10}. We can write

$$\frac{f_3}{f_1} = \frac{f_3}{f_2}\frac{f_2}{f_1} = \frac{g_{ij(3)} \dfrac{1}{-g_{44(3)}} q^i_{(3)} \dfrac{dx^j_{(3)}}{ds_{(3)}}}{g_{ij(2)} \dfrac{1}{-g_{44(2)}} q^i_{(2)} \dfrac{dx^j_{(2)}}{ds_{(2)}}}$$

$$\times \frac{g_{ij(2)} \dfrac{1}{-g_{44(2)}} k^i_{(2)} \dfrac{dx^j_{(2)}}{ds_{(2)}}}{g_{ij(1)} \dfrac{1}{-g_{44(1)}} k^i_{(1)} \dfrac{dx^j_{(1)}}{ds_{(1)}}}$$

$$q^i \overset{\text{def}}{=} \left(\frac{dx^i}{dt}\right)_{\substack{2\to 3\\ \text{photon}}} \quad ; \quad k^i \overset{\text{def}}{=} \left(\frac{dx^i}{dt}\right)_{\substack{1\to 2\\ \text{photon}}} \quad . \tag{7-29}$$

Consider first f_2/f_1:

$$\frac{f_2}{f_1} = \left[\frac{-g_{44(1)} \dfrac{ds_{(1)}}{dt}}{-g_{44(2)} \dfrac{ds_{(2)}}{dt}}\right]\left[\frac{g_{ij(2)}k^i_{(2)} \dfrac{dx^j_{(2)}}{dt}}{g_{ij(1)}k^i_{(1)} \dfrac{dx^j_{(1)}}{dt}}\right] \overset{\text{def}}{=} [\;]_1\,[\;]_2 \quad . \tag{7-30}$$

For the first factor,

$$\frac{ds}{dt} = \left[-g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right]^{1/2} = [-g_{44} - g_{\alpha\beta}v^\alpha v^\beta]^{1/2}$$

$$= [1 + \psi - (1-\psi)\,v^2]^{1/2} = [1 + \psi - v^2]^{1/2} + 0(\psi v^2)$$

where

$$0(\psi v^2) = 0(\frac{M}{r}v^2) = 0(v^4) \sim 10^{-16} \quad .$$

$$\therefore \quad [\;]_1 = \frac{(1+\psi_1)\,[1+\psi_1 - v_1^2]^{1/2}}{(1+\psi_2)\,[1+\psi_2 - v_2^2]^{1/2}}$$

$$[\;]_2 = \frac{1 + \psi_2 - (1-\psi_2)\,\vec{k}_2 \cdot \vec{v}_2}{1 + \psi_1 - (1-\psi_1)\,\vec{k}_1 \cdot \vec{v}_1} = \frac{1+\psi_2}{1+\psi_1} \times \frac{1 - (1-2\psi_2)\,\vec{k}_2 \cdot \vec{v}_2}{1 - (1-2\psi_1)\,\vec{k}_1 \cdot \vec{v}_1} + 0(10^{-20}) \quad . \tag{7-31}$$

So

$$\frac{f_2}{f_1} = \frac{[1+\psi_1 - v_1^2]^{1/2}}{[1+\psi_2 - v_2^2]^{1/2}} \; \frac{1 - (1-2\psi_2)\,\vec{k}_2 \cdot \vec{v}_2}{1 - (1-2\psi_1)\,\vec{k}_1 \cdot \vec{v}_1} \quad . \tag{7-32}$$

It is now convenient to shift to the coordinate system used earlier where the ray travels in the x-y plane and the x-axis is directed along the line between the earth and the planet. Then from Eqs. (6-61) to (6-62):

$$\frac{f_2}{f_1} = \left[\frac{1+\psi_1 - v_1^2}{1+\psi_2 - v_2^2}\right]^{1/2} \frac{1-\left(1+\frac{4M}{r_2}\right)\left[\left(1-\frac{2M}{r_1}\right)v_{2x} + \dot{y}_1(t_2)\, v_{2y}\right]}{1-\left(1+\frac{4M}{r_1}\right)\left[\left(1-\frac{2M}{r_1}\right)v_{1x} + \dot{y}_1(t_1)\, v_{1y}\right]}$$

$$= \left[\frac{1+\psi_1 - v_1^2}{1+\psi_2 - v_2^2}\right]^{1/2} \frac{1 - v_{2x} - \left[\frac{2M}{r_2} v_{2x} + \dot{y}_1(t_2)\, v_{2y}\right]}{1 - v_{1x} - \left[\frac{2M}{r_1} v_{1x} + \dot{y}_1(t_1)\, v_{1y}\right]} + 0(10^{-20})$$

$$= \left[\frac{1+\psi_1 - v_1^2}{1+\psi_2 - v_2^2}\right]^{1/2} \frac{1-v_{2x}}{1-v_{1x}}\left[1 + \frac{2M}{r_1} v_{1x} + \dot{y}_1(t_1)\, v_{1y} - \frac{2M}{r_2} v_{2x} - \dot{y}_2(t_2)\, v_{2y}\right] + 0(10^{-16})$$

$$= \left[\frac{1+\psi_1 - v_1^2}{1+\psi_2 - v_2^2}\right]^{1/2} \frac{1-v_{2x}}{1-v_{1x}} + \frac{2M}{r_1} v_{1x} + \dot{y}_1(t_1)\, v_{1y} - \frac{2M}{r_2} v_{2x} - \dot{y}_2(t_2)\, v_{2y} + 0(10^{-12})$$

$$\stackrel{\text{def}}{=} \left[\frac{1+\psi_1 - v_1^2}{1+\psi_2 - v_2^2}\right]^{1/2} \frac{1-v_{2x}}{1-v_{1x}} + \Delta_{12} \quad . \tag{7-33}$$

Finally, if we perform a similar analysis on f_3/f_2,

$$\frac{f_3}{f_1} = \left[\frac{1+\psi_1 - v_1^2}{1+\psi_3 - v_3^2}\right]^{1/2} \frac{1-\vec{v}_3 \cdot \hat{e}_{23}}{1-\vec{v}_2 \cdot \hat{e}_{23}} \; \frac{1-\vec{v}_2 \cdot \hat{e}_{12}}{1-\vec{v}_1 \cdot \hat{e}_{12}} + \Delta_{12} + \Delta_{23} \tag{7-34}$$

where $\hat{e}_{12}$ and $\hat{e}_{23}$ are unit vectors along the outgoing and returning rays, respectively. The square root can be ignored since

$$\left[\frac{1+\psi_1 - v_1^2}{1+\psi_3 - v_3^2}\right]^{1/2} \approx 1 + \frac{1}{2}(\psi_1 - \psi_3) - \frac{1}{2}(v_1^2 - v_3^2)$$

$$0(\psi_1 - \psi_3) = 0\left(\frac{M}{r_{ES}^2}\,\frac{dr_{ES}}{dt}\, T\right) \simeq 10^{-14} \simeq 0(v_1^2 - v_3^2) \quad .$$

Thus

$$\frac{f_3}{f_1} = \frac{1-\vec{v}_3 \cdot \hat{e}_{23}}{1-\vec{v}_2 \cdot \hat{e}_{23}} \; \frac{1-\vec{v}_2 \cdot \hat{e}_{12}}{1-\vec{v}_1 \cdot \hat{e}_{12}} + \Delta_{12} + \Delta_{23} \tag{7-35}$$

where the first term is just the one predicted by Special Relativity.[63]

If we calculate Δ_{12} and Δ_{23}, they are equal and in fact[64]

$$\Delta_{12} = \Delta_{23} = -\frac{d}{dt}(\delta t)_{out} + 0(\frac{M}{r}\delta t v) \tag{7-36}$$

where from Eqs.(6-105) to (6-109)

$$(\delta t)_{out} = 2M \ln \left[\frac{r_E + r_p + R}{r_E + r_p - R}\right] \qquad \text{[Eq. (6-105)]}$$

$$R = |\vec{r}_E - \vec{r}_p|$$

and for coplanar orbits

$$\frac{d}{dt}(\delta t)_{out} = \frac{2M}{R} \frac{(\dot{r}_E r_p - \dot{r}_p r_E)(r_E - r_p)}{r_E r_p} + \frac{2M}{R}(r_E + r_p)$$

$$\times (\dot{\Theta}_E - \dot{\Theta}_p) \tan[\tfrac{1}{2}(\Theta_E - \Theta_p)] \qquad \text{[Eq. (6-106b)]}$$

which near superior conjunction becomes

$$\frac{d}{dt}(\delta t)_{out} \approx \pm 4M \frac{r_E r_p}{r_E r_p} \frac{(\dot{\Theta}_E - \dot{\Theta}_p)}{d} \quad ; \quad (\pm) \text{ if } \Theta_E - \Theta_p (\lessgtr) \pi \quad . \text{[Eq. (6-108)]}$$

If we treat the earth and the planets as coplanar,

$$\frac{d}{dt}(\delta t)_{out} \approx \left\{\begin{array}{ll} \mp 2.1 \times 10^{-10}\left(\frac{R_o}{d}\right) & \text{Venus} \\ \mp 6.9 \times 10^{-10}\left(\frac{R_o}{d}\right) & \text{Mercury} \end{array}\right\} \quad . \qquad \text{[Eq. (6-109)]}$$

The verification of Eq. (7-36) is straightforward but rather tedious so we will just outline the steps based on Eqs. (6-60) to (6-62) and the definition of Δ_{12} in Eq. (7-32). First we note that

$$\frac{2M}{r_1} v_{1x} - \frac{2M}{r_2} v_{2x} = \frac{2M}{r_E} \vec{v}_E \cdot \hat{e}_{12} - \frac{2M}{r_p} \vec{v}_p \cdot \hat{e}_{12} \quad . \tag{7-37}$$

We now need $\dot{y}_1(t_1)$ and $\dot{y}(t_2)$. Since the y-values are the same at $\vec{r}_E$ and $\vec{r}_p$, Eq. (6-60) tells us that

$$(t_2 - t_1)\left[\dot{y}_1(t_1) + \frac{2M}{d} \frac{x_E}{r_E}\right] = \frac{2M}{d}(r_p - r_E) \tag{7-38}$$

which we approximate to

$$\dot{y}_1(t_1) = \frac{2M}{dR}(r_p - r_E) - \frac{2M}{d} \frac{x_E}{r_E} \quad . \tag{7-39}$$

and then use Eq. (6-62) to obtain

$$\dot{y}_1(t_2) = \frac{2M}{dR}(r_p - r_E) - \frac{2M}{d} \frac{x_p}{r_p} \quad . \tag{7-40}$$

Finally we must express v_y as a dot product, just as we did with v_x. From Fig. 4, we see that the x-axis is in the direction of $\hat{e}_{12}$ while the z-axis is in the direction of $\vec{r}_p \times \vec{r}_E$. Since the x-, y-, and z-axes form a right-handed coordinate system,

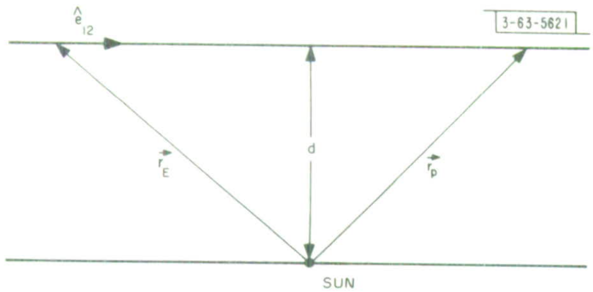

Fig. 4. Geometry of special coordinate system used in discussion of Doppler shift.

$$v_y = \vec{v} \cdot \frac{(\vec{r}_p \times \vec{r}_E) \times \hat{e}_{12}}{|\vec{r}_p \times \vec{r}_E|} \tag{7-41}$$

$$v_x = \vec{v} \cdot \hat{e}_{12} \quad . \tag{7-42}$$

We can now eliminate all reference to our special choice of axes and verify Eq. (7-36).

VIII. SUMMARY

We have tried, in this report, to discuss all those aspects of the General Relativity Theory which can be tested at present or in the near future by performing interplanetary radar experiments and then combining the radar data with data deduced from optical observations to produce ephemerides for the planets. The anomalous time delay and the corrections to the Newtonian equations of motion will certainly be easy to detect, but the corrections to the Doppler shift will not be so easily verifiable.[65]

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REFERENCES

Note: In order to make this report most useful to the reader, references are made to standard monographs rather than to original sources whenever possible.

1. L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Second Edition) (Addison-Wesley, 1962), p. 315.
2. Ibid., Sec. 83.
3. Ibid., Chap. 1.
4. Ibid., Sec. 87.
5. Ibid., Secs. 91-92.
6. R. Adler, M. Bazin and M. Schiffer, Introduction to General Relativity (McGraw-Hill, 1965), Sec. 5.5.
7. J. L. Synge and A. Schild, Tensor Calculus (University of Toronto Press, 1949), Secs. 3.1-3.2.
8. Ref. 1, Secs. 92-93.
9. Ibid., Sec. 95.
10. Ibid., Sec. 32.
11. Ibid., Sec. 33.
12. Ibid., Sec. 37.
13. L. Infeld and J. Plebanski, Motion and Relativity (Pergamon Press, 1960), App. 1.
14. Ref. 1, Sec. 15.
15. Ref. 6, Chap. 9.
16. V. Fock, "Three Lectures on Relativity Theory," Rev. Mod. Phys. 29, 325-333 (1957).
17. V. Fock, The Theory of Space, Time and Gravitation (Pergamon Press, 1959).
18. L. I. Schiff, "Motion of a Gyroscope According to Einstein's Theory of Gravitation," Proc. Nat. Acad. Sci. U. S. A., 46, 871-882 (1960).
19. Ref. 13, Sec. 1.6.
20. Ref. 1, Sec. 105.
21. P. G. Bergmann, Introduction to the Theory of Relativity (Prentice-Hall, 1942), pp. 180-187.
22. F. A. E. Pirani, "Introduction to Gravitational Radiation Theory" in Lectures on General Relativity (Prentice-Hall, 1965), pp. 249-374.
23. Ref. 1, Sec. 87.
24. Ibid., Sec. 96.
25. P. G. Bergmann, "The General Theory of Relativity" in Handbuch der Physik, Vol. IV, Springer-Verlag, Berlin (1962), pp. 247-255.
26. Ibid., p. 331.
27. Ibid., Sec. 97.
28. J. L. Anderson and R. Gautreau, "Possible Causal Violations at Radii Greater Than the Schwarzschild Radius," Physics Letters 20, 24-25 (1966).
29. A. S. Eddington, The Mathematical Theory of Relativity (Cambridge University Press, 1924), p. 97.
30. F. Rohrlich, Classical Charged Particles (Addison-Wesley, 1965), Chap. 6.
31. Ref. 1, Sec. 65.

32. Ref. 16, pp. 67-79.

33. A. I. Akhiezer and V. B. Berestetsky, Quantum Electrodynamics, Moscow (1953) (AEC Translation 2784), pp. 385-395.

34. Ref. 1, Sec. 104.

35. I. Carmeli, "The Equations of Motion for Slowly Moving Particles in the General Theory of Relativity," Nuovo Cimento 37, 842-875 (1965).

36. A. Trautman, "Foundations and Current Problems of General Relativity" in Ref. 22, pp. 1-248 (cf. pp. 127-130).

37. Ref. 1, Sec. 101.

38. Ref. 13, Chap. 3.

39. Ref. 11, pp. 407-410.

40. Ref. 17, App. B and Sec. 53.

41. L. Infeld, "Equations of Motion in General Relativity Theory and the Action Principle," Rev. Mod. Phys. 29, 398-411 (1957).

42. Ref. 21, Chap. 15.

43. Ref. 42, pp. 175-179.

44. Ref. 25, pp. 242-245.

45. Ref. 13, Chap. 1.

46. A. S. Eddington and G. L. Clark, "The Problem of N Bodies in General Relativity Theory," Proc. Roy. Soc. A166, 465-475 (1938).

47. W. De Sitter, "On Einstein's Theory of Gravitation and Its Astronomical Consequences II," Mon. Nat. R. Astr. Soc. 77, 155-184 (1917).

48. Ref. 13, Sec. 3. 4.

49. Ref. 17, Sec. 81.

50. Ref. 29, Sec. 44.

51. J. L. Synge, "Introduction to General Relativity" in Relativity, Groups, and Topology (Gordon and Breach, 1964), Sec. 1. 6.

52. E. Schrodinger, Expanding Universes (Cambridge University Press, 1956), Sec. 7.

53. Ref. 1, p. 338.

54. Ibid., Sec. 5.

55. I. I. Shapiro, "Radar Determination of the Astronomical Unit," Bulletin Astronomique 25, 177-215 (1965).

56. Ref. 1, Sec. 84.

57. M. Ash and I. I. Shapiro, "Refinement of Planetary and Lunar Ephemerides," to be published as Technical Report, Lincoln Laboratory, M. I. T.

58. Ref. 36, Sec. 6. 5.

59. Ref. 1, Sec. 9.

60. Ref. 52, pp. 47-50.

61. Ref. 36, Sec. 6. 10.

62. Ref. 52, pp. 43-44.

63. Ref. 55, Eq. (5-5).

64. I. I. Shapiro, "Fourth Test of General Relativity," Phys. Rev. Letters 13, pp. 789-791 (1964).

65. I. I. Shapiro, "Effects of General Relativity on Interplanetary Time-Delay Measurements," Technical Report 368, Lincoln Laboratory, M. I. T. (1964), DDC 614232.

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13. ABSTRACT

A discussion is presented of all aspects of the General Relativity Theory that can be tested at present or in the near future by performance of interplanetary radar experiments. The assumptions underlying general relativity and the mathematical concepts and techniques used are first introduced. Then the approximate equations of motion, to the accuracy needed for the solar system, are derived. Finally, the propagation of electromagnetic waves according to general relativity is considered, and expressions for the time delays and Doppler shifts are obtained.

14. KEY WORDS

general relativity theory   Newtonian gravitational theory   interplanetary radar experiments